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Chapter 1: INEQUALITIES

§ 1.1 Inequations :

We know how to find the solution-set of an equation. An equation is an open sentence in one or more variables. Any real number which on replacing a variable makes this open statement a true statement is known as a solution of the equation. The set of all such solutions is known as the solution-set or the truth-set of the given equation. While solving equations we make use of the properties of real numbers and also the properties of the relation of equality. Using those properties we reduce the given equation into simpler but equivalent equation and finally arrive at the solution-set.

The verb in an open sentence which is an equation is denoted by the sign of equality, viz. ' $=$ '. In some open sentences the verb may be denoted by other familiar signs such as ' $>$ ', ' $<$ ' or ' $\geq$ ', ' $\leq$ '. Open sentences with these signs are known as **inequalities** or **inequations**. To solve an inequation means to find the set of values of the variable or variables which make the given inequation a true statement. That is to say, to solve an inequation is the same as to determine its solution-set.

§ 1.2 Properties of Order :

For finding solution-sets of inequations we have to rely on the **properties of order** in real numbers and obtain simpler but equivalent inequations. Let us, therefore, revise the properties of the order in real numbers.

If a, b, c are real numbers, and

- (1) if $a < b$, then $(-a) > (-b)$ (order in additive inverses)
- (2) if $a < b$, then $(a + c) < (b + c)$, and $(a - c) < (b - c)$. } ... (addition property of order)
- (3) if $(a + c) < (b + c)$, then $a < b$ (cancellation law for addition)
- (4) if $a < b$, and $c > 0$, then $ac < bc$,
and if $a < b$ and $c < 0$, then $ac > bc$. } ... (multiplication property of order)
- (5) if $ac < bc$ and $c > 0$, then $a < b$
and if $ac < bc$ and $c < 0$, then $a > b$. } ... (cancellation law for multiplication)
- (6) if $a < b$ and $b < c$, then $a < c$ (transitive property of order)

Let us see how we can employ the above properties in finding the solution-sets of given inequations.

Example 1: Find the solution-set of $2x + 9 < 5x - 6$.

Adding $(-5x)$ to both the sides of the given inequation we get,

$$(-5x) + 2x + 9 < (-5x) + 5x - 6$$

$$\therefore -3x + 9 < -6 \quad (\text{associativity and identity for addition})$$

Adding (-9) to both the sides, $-3x + 9 + (-9) < -6 + (-9)$

$$\therefore -3x < -15 \quad (\text{associativity and identity for addition})$$

Multiplying both the sides by $(-\frac{1}{3})$, $(-\frac{1}{3})(-3x) > (-\frac{1}{3})(-15)$

(Note.—Multiplying the two sides of an inequation by a negative number reverses the sign of the inequation.)

$$\therefore x > 5.$$

$\therefore$ The required solution-set is the set of all real numbers that are greater than 5.

This solution set is written as

$$\{x \mid x \in \mathbb{R}, x > 5\}$$

which is read as 'the set of all x 's such that x is a real number and greater than 5.'

[While writing the above solution-set we have implicitly assumed that the replacement set of x is the set of real numbers. It is obligatory to state the replacement set of a variable or variables that appear in open sentences. Generally, we are dealing with real numbers and we tacitly assume that the replacement set is the set of real numbers. When the replacement set is other than that of real numbers, it has to be explicitly stated.]

While solving the above inequation we made use of some of the properties of order. Operations done on the basis of the properties of order do not change the solution-set of the original inequation. From this fact we obtained simpler but equivalent inequations and finally arrived at the simplest inequation ' $x > 5$ ', which gave the required solution-set.

The solution-sets of ' $2x + 9 < 5x - 6$ ' and ' $x > 5$ ' are the same. Let us verify. Substituting 6 for x , a number greater than 5 in the original inequation, the left-hand side of the inequation reduces to 21, while the right-hand side reduces to 24. Because $21 < 24$, we get a true statement by the said replacement.

This would happen for any number greater than 5.

Example 2: (i) Taking the set of natural numbers as the replacement-set for x and (ii) taking the set of real numbers as the replacement-set for x , find the solution-sets of $4x - 7 < 2x + 5$, and draw their graphs.

$$4x - 7 < 2x + 5$$

$$\therefore 4x - 7 + 7 < 2x + 5 + 7 \quad (?)$$

$$\therefore 4x < 2x + 12 \quad (?)$$

$$\therefore 4x - 2x < 2x + 12 - 2x \quad (?)$$

$$\therefore 2x < 12 \quad (?)$$

$$\therefore x < 6 \quad (?)$$

$$\therefore \text{The Solution-set of } '4x - 7 < 2x + 5' \text{ is } \{x \mid x < 6\}$$

(i) The solution-set, when the replacement-set is the set of natural numbers, becomes $\{x | x \in \mathbb{N}, x < 6\}$ which is the same set as $\{1, 2, 3, 4, 5\}$.

(ii) When the replacement-set for x is taken to be the set of real numbers, the solution-set cannot be listed, but which contains all real numbers less than 6. As this set cannot be listed we show it as $\{x | x \in \mathbb{R}, x < 6\}$.

The graphs of the two solution-sets are shown below.

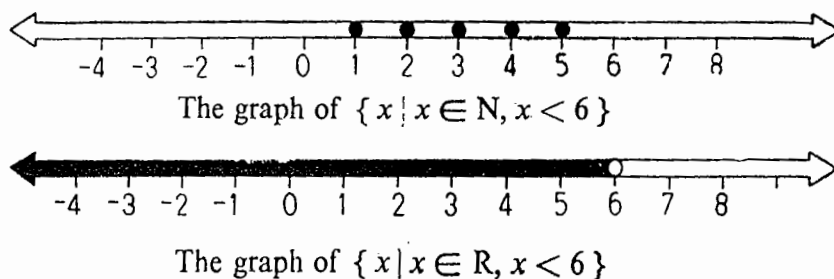


Fig. 1.1

§ 1.3 Compound Inequations :

The two open sentences ' $3 < x$ ' and ' $x < 5$ ' can be combined together and written in a shorter form as ' $3 < x < 5$ '. This combined sentence means that x is greater than 3 and less than 5.

To combine the two sentences ' $x > -5$ ' and ' $x < 8$ ' you must notice that ' $x > -5$ ' means the same thing as ' $-5 < x$ '. So the above two sentences can be written together as ' $-5 < x < 8$ '.

The two sentences in each of the above examples are joined by the word **and**.

The open sentence ' $3 < x$ ' implies the set of all numbers greater than 3, while ' $x < 5$ ' represents the set of all numbers less than 5. Therefore, any real number which satisfies the two conditions—(i) it is less than 5, (ii) it is greater than 3—is a solution of the compound sentence $3 < x < 5$. When two open sentences are joined by the word **and**, the solution-set of the resulting open sentence denotes the **intersection** of the two sets cited above. This is made clear in the following diagram.

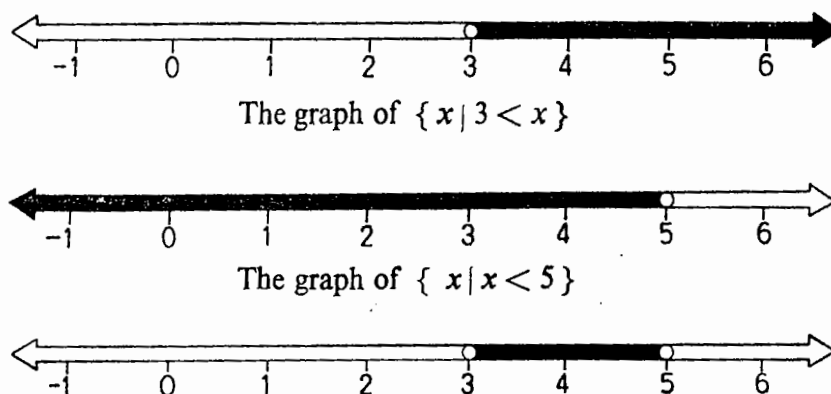


Fig. 1.2

EXERCISE 1.1

Combine each of the pairs of following open sentences into a single sentence. Draw graphs of each and also the graph of each of the compound open sentences.

1. $2 < x, x < 6$
2. $x > -3, x < 1$
3. $x < 3.5, x > 1$
4. $x > -4, x < -2$.

Sometimes two or more sentences are joined by the word **or**. How can the two open sentences ' $x > -3$ ' and ' $x < 3$ ' be expressed in a single open sentence which will denote that they are joined by the word **or**?

Recollect the definition of the 'absolute value' of a number. We have defined the absolute value of x as follows.

- (i) When $x > 0$, $|x| = x$,
- (ii) When $x = 0$, $|x| = x$,
- (iii) When $x < 0$, $|x| = -x$.

So, $|x| = 2$ means that when x is positive, then $x = 2$, **or** in case x is negative, then $-x = 2$. ' $-x = 2$ ' is the same as ' $x = -2$ '.

Similarly, $|x| < 3$ means ' $x < 3$ **or** $-x < 3$ ', which means the same as ' $x < 3$ **or** $x > -3$ '.

The above is one type in which we can express two sentences joined by the word **or**, in a single sentence. Another type obtained is as in the following sentence—

$$(x - 3)(x + 2) = 0.$$

This sentence means, 'either $x - 3 = 0$ **or** $x + 2 = 0$ '. In the same way, we can join two similar inequality statements.

Sometimes, we can combine one equality statement with one inequality statement. For instance, the two open sentences ' $x > 3$ ' **or** ' $x = 3$ ' are combined in a single open sentence as ' $x \geq 3$ '. Similarly we write ' $2x - 3 \leq 5$ ' for the two open sentences ' $2x - 3 = 5$ ' **or** ' $2x - 3 < 5$ '.

EXERCISE 1.2

Write each of the following pairs of open sentences as a single open sentence.

1. $x = 2$ **or** $x = -2$
2. $x > 2$ **or** $x < -2$
3. $-3 > x$ **or** $x > 3$
4. $x - 2 = 0$ **or** $x + 3 = 0$
5. $x + 5 = 0$ **or** $2x - 6 = 0$
6. $x = 4$ **or** $x - 5 = 0$
7. $x < 3$ **or** $x = 3$
8. $2x + 3 > 8$ **or** $2x + 3 = 8$.

§ 1.4 Solution Sets of Compound Sentences :

Example 3 : Solve. $7x - 1 > 3x + 1 > 8$.

The two parts of the given open sentence are joined by the word 'and' and they are

$$\begin{array}{ll} 7x - 1 > 3x + 1 & \text{..... (1)} \\ \text{and} & 3x + 1 > 8 \quad \text{..... (2)} \end{array}$$

To find the solution-set of the compound inequation is to find the set of replacements for x that will make **both** the open sentences $7x - 1 > 3x + 1$ and $3x + 1 > 8$, true.

Let us solve the first part :

$$7x - 1 > 3x + 1$$

Adding $-3x$ to both sides, $4x - 1 > 1$

Adding 1 to both sides, $4x > 2$

Multiplying both sides by $\frac{1}{4}$, $x > \frac{1}{2}$ (I)

Now, let us turn to the second part :

$$3x + 1 > 8$$

Adding -1 to both sides, $3x > 7$

Multiplying both sides by $\frac{1}{3}$, $x > \frac{7}{3}$ (II)

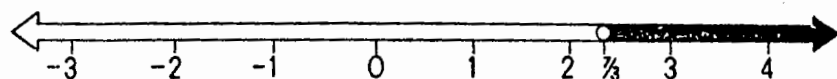
$\therefore$ From (I) and (II), the solution-sets of the two parts are

$\{x | x > \frac{1}{2}\}$ and $\{x | x > \frac{7}{3}\}$ respectively. The intersection-set of the two sets is the solution-set of the given compound open sentence, which is $\{x | x > \frac{7}{3}\} \cap \{x | x > \frac{1}{2}\}$

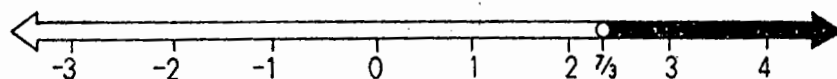
The above solution-set means that 'x is a number greater than $\frac{1}{2}$ as well as greater than $\frac{7}{3}$ '. The two conditions can be satisfied only if x is greater than $\frac{7}{3}$. Hence, the solution-set of the compound open sentence can be written as $\{x | x > \frac{7}{3}\}$. In the following diagram, graphs of the solution-sets of the two parts are shown on different number-lines and then the graph of the joint solution-set is shown.



The graph of $\{x | x > \frac{1}{2}\}$



The graph of $\{x | x > \frac{7}{3}\}$



The graph of $\{x | x > \frac{1}{2}\} \cap \{x | x > \frac{7}{3}\}$

Fig. 1.3

Example 4 : Find the solution-set of the open sentence $-7 < 3x + 2 < x + 5$ and draw its graph.

(1) $-7 < 3x + 2$

Adding -2 to both sides, $-9 < 3x$

Multiplying both sides by $\frac{1}{3}$, $-3 < x$ (I)

(2) $3x + 2 < x + 5$

Adding $-x$ to both sides, $2x + 2 < 5$

Adding -2 to both sides, $2x < 3$

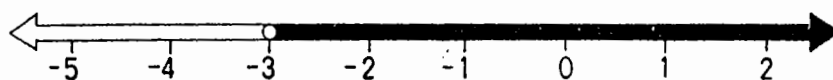
Multiplying both sides by $\frac{1}{2}$, $x < \frac{3}{2}$ (II)

From (I) and (II), the solution-set of the given open sentence is

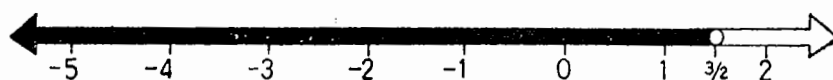
$$\{x | -3 < x\} \cap \{x | x < \frac{3}{2}\},$$

which is the same as $\{x | -3 < x < \frac{3}{2}\}$.

In the figure below graphs of the two parts are shown separately and then the graph of the intersection-set is shown.



The graph of $\{x | -3 < x\}$



The graph of $\{x | x < \frac{3}{2}\}$



The graph of $\{x | -3 < x < \frac{3}{2}\}$

Fig. 1-4

Example 5 : Solve and draw the graph of the solution-set of $|x + 1| > 5$.

(1) When $(x + 1)$ is positive, its absolute value is $(x + 1)$.

(2) When $(x + 1)$ is negative, its absolute value is $-(x + 1)$.

$\therefore$ (1) When $(x + 1)$ is positive, $x + 1 > 5$ (I)

or (2) When $(x + 1)$ is negative, $-(x + 1) > 5$ (II)

The solution-set of the compound sentence will contain numbers which satisfy the condition (I) or the condition (II). In this case, note that a real number cannot satisfy both the conditions at the same time because of the law of trichotomy. So that, it is therefore enough that a real number in this case should satisfy at least one of the conditions (I) or (II). So, the solution-set of the combined open sentence is the **union-set** of the two solution-sets of the open sentences $x + 1 > 5$ and $-(x + 1) > 5$.

$$(1) \quad x + 1 > 5$$

Adding -1 to both sides, $x > 4$ (I)

$$(2) \quad -(x + 1) > 5$$

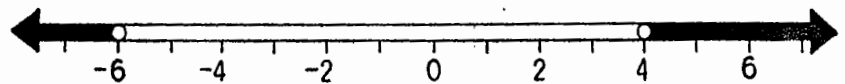
Multiplying both sides by (-1) , $x + 1 < -5$.

(Note.—Multiplying both sides by a negative number reverses the sign of the inequality.)

Adding -1 to both sides, $x < -6$ (II)

$\therefore$ From (I) and (II), the solution-set of the given compound open sentence is $\{x | x > 4 \text{ or } x < -6\}$

The graph of this solution-set is shown below.



The graph of $\{x | x > 4 \text{ or } x < -6\}$

Fig. 1.5

Example 6: Find and draw the graph of the solution set of $(x - 3)(x + 2) > 0$

We know, that when the product of two numbers is positive, either both of them should be positive or both of them should be negative.

$\therefore$ The given open sentence means

(1) $x - 3 > 0$ and $x + 2 > 0$.

or (2) $x - 3 < 0$ and $x + 2 < 0$.

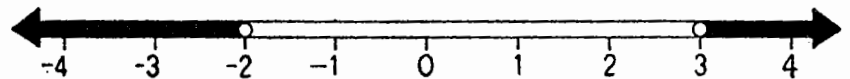
(1) The solution-set containing numbers satisfying both the conditions in (1) above viz, $x > 3$ and $x > -2$ is $\{x | x > 3\}$.

(2) The solution-set containing numbers satisfying both the conditions $x < 3$ and $x < -2$ is $\{x | x < -2\}$.

The union-set of these two solution-sets is the solution-set of the given combined open sentence, which is

$\{x | x > 3\} \cup \{x | x < -2\}$ which is written as $\{x | x > 3 \text{ or } x < -2\}$.

The graph of this solution-set is shown below.



The graph of $\{x | x > 3 \text{ or } x < -2\}$

Fig. 1.6

Example 7: Find the solution-set of $x^2 + 2x - 3 < 0$ and draw its graph.

$x^2 + 2x - 3 < 0$ can be written as $(x + 3)(x - 1) < 0$.

Here the product of $(x + 3)$ and $(x - 1)$ is negative. Hence one of them must be positive and the other negative. The possible cases are

(1) $x + 3 > 0$ and $x - 1 < 0$;

or (2) $x + 3 < 0$ and $x - 1 > 0$.

(1) The set of values of x satisfying both the conditions in (1) above is $\{x | -3 < x < 1\}$.

(2) The set of values of x satisfying both the conditions in (2) above is empty, since there are no values of x such that $x < -3$ and also $x > 1$.

$\therefore$ The solution-set of the given compound open sentence is $\{x | -3 < x < 1\}$.

Instead of proceeding as in the previous discussion we may proceed to find the solution-set from the following table. We want values of x such that $(x + 3)(x - 1)$ is negative.

x	$x + 3$	$x - 1$	$(x + 3)(x - 1)$
$x < -3$	negative	negative	positive
$x = -3$	0	negative	0
$-3 < x < 1$	positive	negative	negative
$x = 1$	positive	0	0
$x > 1$	positive	positive	positive

The condition that $(x + 3)(x - 1) < 0$ is satisfied by values of x ranging from -3 to 1 (both values being excluded). Hence, the solution-set is—

$$\{x \mid -3 < x < 1\}.$$

The graph of the solution-set is shown below.



The graph of $\{x \mid -3 < x < 1\}$

Fig. 1.7

Example 8: If the open sentence in example 7 above is modified as—
 $x^2 + 2x - 3 \leq 0$, see how the solution-set and its graph are changed.
 $x^2 + 2x - 3 \leq 0$ is split-up into two parts as follows.

$$(1) x^2 + 2x - 3 < 0 \quad \text{or} \quad (2) x^2 + 2x - 3 = 0.$$

The first part has been dealt with in example 7 above. In addition we have to consider the solution-set of (2).

$$x^2 + 2x - 3 = 0 \quad \therefore (x + 3)(x - 1) = 0$$

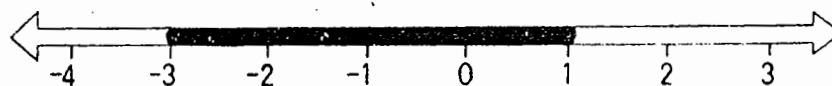
$$\therefore x + 3 = 0 \quad \text{or} \quad x - 1 = 0$$

$$\therefore x = -3 \quad \text{or} \quad x = 1$$

Hence, the solution-set of the second part is $\{x \mid x = -3 \quad \text{or} \quad x = 1\}$

The solution-set of the combined open sentence must contain values of x such that the product of $(x + 3)(x - 1)$ is either negative or zero. From the table prepared in example 7 above, we see that the values of x satisfying the above conditions are $x = -3$, $-3 < x < 1$, $x = 1$. All these values of x can be represented by the set $\{x \mid -3 \leq x \leq 1\}$.

The graph of this solution-set differs from the graph of the solution-set in example 7. In Fig. 1.8 the end-points of the segment, representing -3 and 1 are included and hence they are shown with the thick ink.



The graph of $\{x \mid -3 \leq x \leq 1\}$

Fig. 1.8

EXERCISE 1.3

Write the solution-sets of the following inequations by mere inspection.

1. $x - 5 > 2$ 2. $2 < x + 1$ 3. $3x < 9$ 4. $3x > 2x - 5$
5. $3x \leq 12$ 6. $-4x < 20$ 7. $7x \geq -23 + x$ 8. $-8x - 3 < -2x$

Find and draw the graphs of the solution-sets of the following.

9. $4x + 1 > 7x - 5$ 10. $7x - 6 \geq 4 + 17(x - 5)$
11. $\frac{4x + 3}{3} > x + 2$ 12. $2(x + 3) \geq 5(x - 4)$
13. $|5x - 15| > 0$ 14. $|3x + 4| > 0$

Find the truth-sets of the following.

15. $|4x - 5| < 2$ 16. $\left|4 - \frac{x}{7}\right| \leq 10$
17. $11 \leq 3x + 5 \leq 17$ 18. $-3 \leq -x + 2 < 0$

State whether the values of the expressions are positive, negative or zero, under the given conditions.

19. $(x + 3)(x - 1)$, when
 - (i) $x = -3$, (ii) $x < -3$, (iii) $x = 1$,
 - (iv) $x > 1$, (v) $x = 0$, (vi) $x = -2$.
20. $(x - 2)(x - 5)$, when
 - (i) $x = 2$, (ii) $x < 2$, (iii) $x = 5$,
 - (iv) $x > 5$, (v) $x = 3$, (vi) $x = 4.5$

Find the truth-sets of the following and draw their graphs.

21. $(x + 2)(x - 2) > 0$ 22. $(x - 1)(x - 4) < 0$
23. $x(x + 5) \geq 0$ 24. $x^2 - 16 > 0$ 25. $8 > 2x + x^2$
26. When 8 is added to a certain natural number, the result is less than three times the number. Find such natural numbers.
27. When 20 is subtracted from four times a natural number, the result is less than the number. Find all such natural numbers.
28. A wire 18 cm long is shaped into a rectangle, such that the measures of its sides will be whole numbers and the area will be maximum. Find the measures of the sides of the rectangle with greatest area.
29. Complete the following statements.
If $a < 0$ and $b < 0$, then $a + b$ is _____. (positive, negative or zero).
30. If $x > 0$ and $y < 0$, and $|x| > |y|$, then $x + y$ is _____. (positive, negative or zero).
31. If $x > 0$ and $yx < 0$, then y is _____. (positive, negative or zero).
32. State whether the graphs of the truth-sets of each of the following open sentences is (i) a line, (ii) a ray, (iii) a segment or (iv) a point.
(i) $x - 2 \leq 7$; (ii) $2x + 3 \geq 7$; (iii) $2x - 3 \leq x$; (iv) $-4 \leq x \leq 2$.

Chapter 2: CARTESIAN PRODUCT

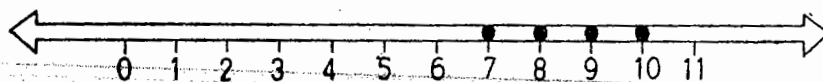
§ 2.1 Graphs of Sets of Numbers :

You know what is meant by a graph of a set of numbers. You also know how to draw the graphs of the solution sets of inequalities in one variable (See art. 3, Chap. 5, Algebra Std. IX). We had also some practice of drawing graphs of the solution sets of inequalities. (See Chap. 1.) Let us revise some of the important points regarding graphs and how to draw them.

The number which denotes a point on the number-line is, as you know, called the **co-ordinate** of the point and the point is called the **graph** of the number.

Let ' x ' be a member of the set A and let ' x ' be greater than 6. All such ' x 's must be the members of the set A .

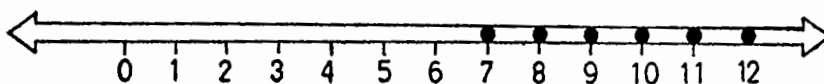
Example 1: Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. The graph of $\{x \in A, x > 6\}$ is the set of points shown in bold dots in fig. 2.1 below.



The graph of $\{x \in A, x > 6\}$

Fig. 2.1

Example 2: If the replacement set of ' x ' be the set of natural numbers, then the graph of $\{x \in \mathbb{N}, x > 6\}$ will contain all points on the number-line denoting natural numbers greater than or equal to 7.



The graph of $\{x \in \mathbb{N}, x > 6\}$

Fig. 2.2

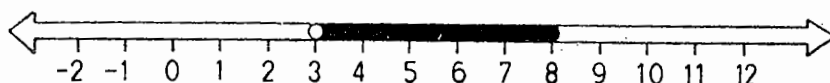
Example 3: If the replacement set of ' x ' be the set of rational numbers, the graph of $\{x \in \mathbb{Q}, 3 < x < 8\}$ will be as shown in fig. 2.3 below. The graph contains all points denoting rational numbers between 3 and 8, and hence the portion of the number-line between 3 and 8 contains innumerable points which do not represent rational numbers. The replacement set does not contain the irrational numbers. Hence, the graph is shown as a broken open segment. The segment is open as the numbers 3 and 8 are excluded in the given set and hence those two points are shown with hollow circles.



The graph of $\{x \in \mathbb{Q}, 3 < x < 8\}$

Fig. 2.3

Example 4 : Let the replacement set of 'x' be the set of real numbers. Then the graph of $\{x \in \mathbb{R}, 3 < x \leq 8\}$ would contain all the points representing real numbers between 3 and 8 and as every point on the number-line represents a unique real number, all the points between 3 and 8 are included in the graph. Hence, the graph is shown by a bold continuous segment. Moreover, as 3 is excluded from the given set, while 8 is included in it, the point representing 3 is marked with a hollow circle while the point representing 8 is shown with a thickly shaded circle.



The graph of $\{x \in \mathbb{R}, 3 < x \leq 8\}$

Fig. 2.4

EXERCISE 2.1

Draw the graphs of the following sets using a separate number-line for each.

1. $\{x \in \mathbb{N}, x < 5\}$
2. $\{x \in \mathbb{N}, x > 5\}$
3. $\{x \in \mathbb{I}, x < 5\}$
4. $\{x \in \mathbb{I}, x \leq -5\}$
5. $\{x \in \mathbb{N}, 2 < x < 8\}$
6. $\{x \in \mathbb{I}, -3 \leq x < 5\}$
7. $\{x \in \mathbb{Q}, x > -3\}$
8. $\{x \in \mathbb{R}, x > -3\}$
9. $\{x \in \mathbb{Q}, -2.5 \leq x \leq 3.5\}$
10. $\{x \in \mathbb{R}, -2.5 < x < 3.5\}$

Describe the sets of numbers shown in the graphs for examples 11 to 15 below.

(A single number-line is shown at the bottom for all the graphs.)

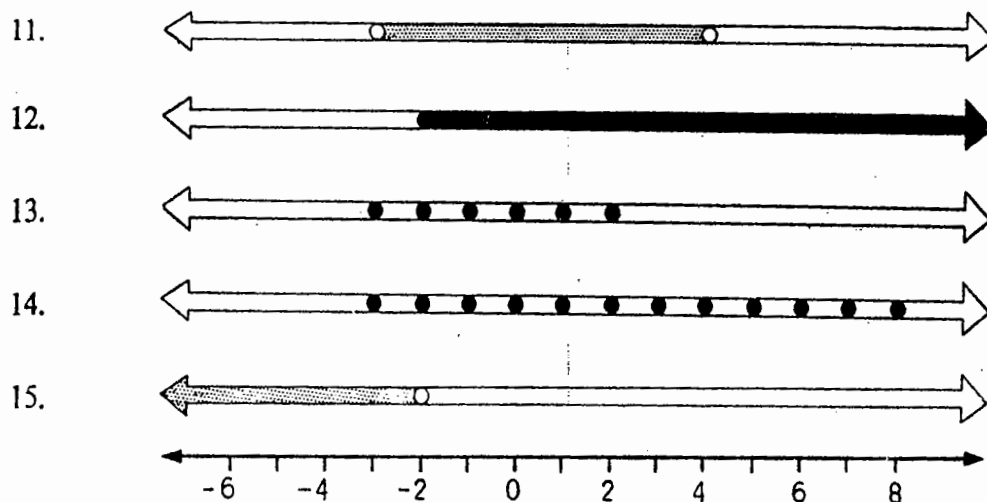


Fig. 2.5

§ 2.2 Ordered Pair :

We have seen that points on a line represent numbers. It is but natural to know what figures are represented by sets of points on a plane. First we must understand the meaning of the term 'an ordered pair'.

Suppose, you are asked, "Where do you sit in your class?" and you reply, "In the fourth row from the left." Would your reply make me understand as to where exactly your seat is?

If your answer is, "In the fourth row from the left and in the third row from the front", I would still not know where you sit. Your answer does not make it clear whether the seats are to be counted from right or from left.

The diagram below represents a map of the seats in a class. Each seat is marked with two numbers. The first number shows the column and the second number represents the number of the row, in which the seat is located—

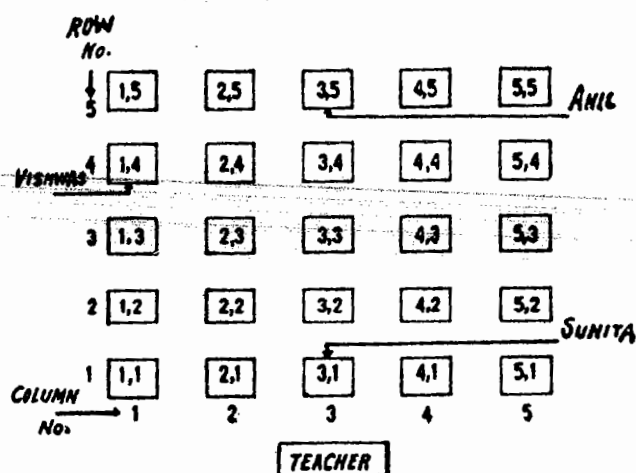


Fig. 2.6

If we say, "Anil's seat is in the third column", it would not be enough to locate the seat of Anil. The third column contains 5 seats. So, if we add the information "Anil sits in the fifth row", it would be enough to determine the seat on which Anil sits in the class.

With reference to the above map, the seats of Anil, Sunita and Vishwas can be shown by the pairs of numbers (3, 5), (3, 1), (1, 4) respectively.

We cannot describe Anil's seat as (5,3) instead of (3, 5). Why? We have decided to state the number of the column first and then the number of the row. So, the pairs of numbers (3, 5) and (5, 3) represent different seats. According to the above 'convention' (decision) the seats can be distinctly represented by pairs of numbers. There is a definite but different meaning to the first and the second elements of the above pairs of numbers. Hence, the order of the numbers in those pairs does matter.

Instead of numbering the seats with pairs of numbers we may use two reference lines with units marked on them and represent the map in Fig. 2.6 as shown in the following diagram.

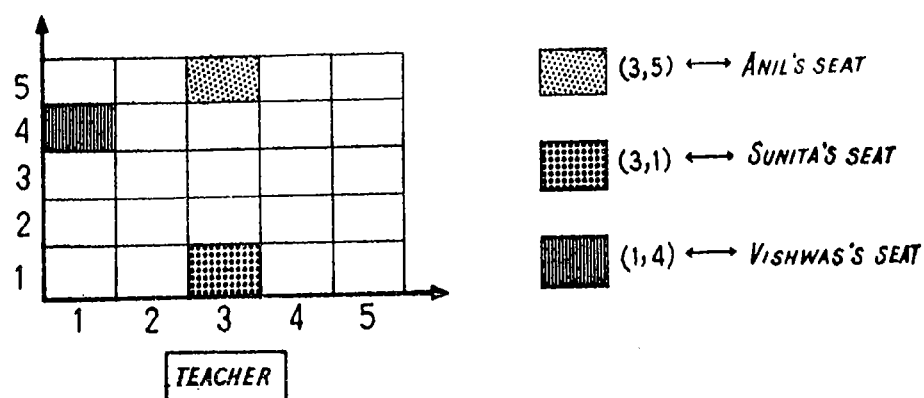


Fig. 2.7

$\{3, 5\}$ means the set whose members are the numbers 3 and 5. The set $\{5, 3\}$ is not different from the set $\{3, 5\}$ as the order in which the members of a set are listed does not matter. But the order of numbers in the pairs (3, 5) and (5, 3) is important. So we name such a pair of numbers an **ordered pair**.

We discussed above how ordered pairs of numbers can be used to specify the seat in a class. The elements of an ordered pair need not be numbers only. Ordered pairs are used to denote different seats in a theatre, as you may be knowing. If your ticket bears the ordered pair (C, 5), it means that your seat is the fifth seat in row 'C'. In this example you see that the first member of the ordered pair is not a number, but a letter of the alphabet.

We use the notion of ordered pairs on many occasions. Ordered pairs are used to tell the score in a volley-ball or table-tennis game. When the referee shouts (11, 8) at a certain time during a game, it means the player or the team who is delivering the service has scored 11 points while his opponent or the opponent team has scored 8 points. You must have observed that when the score reaches (11, 9) the service is changed and the same score is repeated as (9, 11). This fact should bring to you notice the importance of the order of the elements in an ordered pair.

In your geographical atlas book there is an index of names of cities wherein two numbers denoting the latitude and the longitude are shown against each city. It is the convention to state the latitude of a place first and then the longitude. So the meaning of the first number in an ordered pair denoting latitude and longitude of a place has quite a different meaning from that of the second number.

The above examples stress the fact, that

the places of the elements in an ordered pair have different meanings
and hence the order of the elements of an ordered pair is important.

§ 2.3 Cartesian Product :

Let $A = \{ 1, 3, 5, 7 \}$ and $B = \{ 2, 4, 6, 8 \}$.

How many ordered pairs with the first member from the set A and the second member from the set B can be formed?

We can associate a member of the set B to each member of the set A in four ways. As the set A contains 4 members we should have $4 \times 4 = 16$ ordered pairs.

Make a table as in the following diagram and complete it.

		Element from B			
		2	4	6	8
Element from A	1	(1,2)	(1,4)	(1,6)	(1,8)
	3	(3,2)	(3,4)	(3,6)	(3,8)
	5				
	7				

← ordered pairs with 1 from set A associated with each member of set B.

← ordered pairs with 3 from set A associated with each member of set B.

We can form a set of all the sixteen ordered pairs formed from the elements of the given sets A and B. This set is represented by the symbol $A \times B$ (read as, A cross B) and is called the **Cartesian Product** of the sets A and B.

Thus, $A \times B = \{ (1, 2), (1, 4), (1, 6), (1, 8), (3, 2), (3, 4), (3, 6), (3, 8), (5, 2), (5, 4), (5, 6), (5, 8), (7, 2), (7, 4), (7, 6), (7, 8) \}$

It is not necessary that the elements of ordered pairs be from two different sets. The ordered pairs representing the seats in a class or the scores of two opponent teams in a game of table-tennis, or the latitude and the longitude of a place, fully support the above statement. If both the elements of ordered pairs are taken from the whole set of natural numbers N, then the Cartesian Product Set $N \times N$ will be an infinite set.

$$N \times N = \{ (1, 1), (1, 2), (1, 3), \dots, (2, 1), (2, 2), (2, 3), \dots, (5, 3), (5, 4), (5, 5), \dots, (437, 101), (437, 102), \dots \}$$

In the product set $I \times I$ (which again is an infinite set), both the elements of the ordered pairs will be integers.

There will be infinite pairs such as $(-5, 3), (-5, -4), (-5, 0), (-10, 342), (24, -7), (-24, -3856), (6, 0), (0, 0), (0, -8), \dots$ in the said set.

If both the elements of the ordered pairs are real numbers, we denote the set of such ordered pairs ' $R \times R$ ', which is also an infinite set of ordered pairs.

EXERCISE 2.2

1. If $A = \{ 1, 2, 3 \}$, $B = \{ 1, 2, 3, 4 \}$, how many ordered pairs will the cartesian product set $A \times B$ contain? Give reasons.
2. Write the set $A \times B$ in Ex. 1 listing the ordered pairs.
3. If $C = \{ a, b, c, d \}$ and $D = \{ 1, 2, 3, 4, 5 \}$ list the sub-set of $C \times D$ in which the first element of the ordered pairs is 'b'.

4. List the sub-set of $C \times D$ in Ex. 3, in which the second element of the ordered pairs is 2 or 5.
5. When do we say that two sets are equal? Does the same definition hold for two ordered pairs to be equal? If $(x, 5)$ and $(6, y)$ are equal ordered pairs, then $x = 6$ and $y = 5$. Determine x and y in the following equations :
 (i) $(x - 1, y + 2) = (1, 0)$; (ii) $(2x - 1, y - 5) = (7, 2y + 1)$

§ 2.4 Points on a Plane and Ordered Pairs :

Every point on the number-line represents the graph of some unique real number. Take a number-line on your note-book. Mark a point P in the plane of the leaf of your note-book, so that it does not lie on the line.

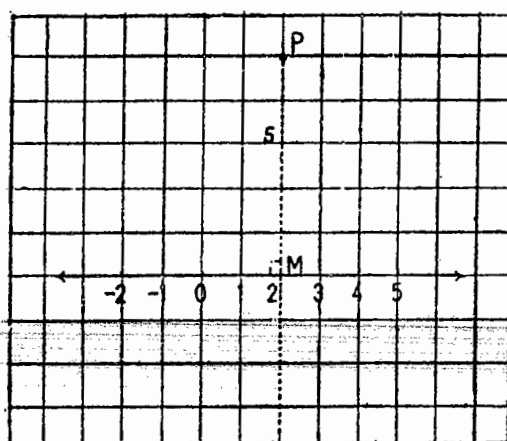


Fig. 2.8

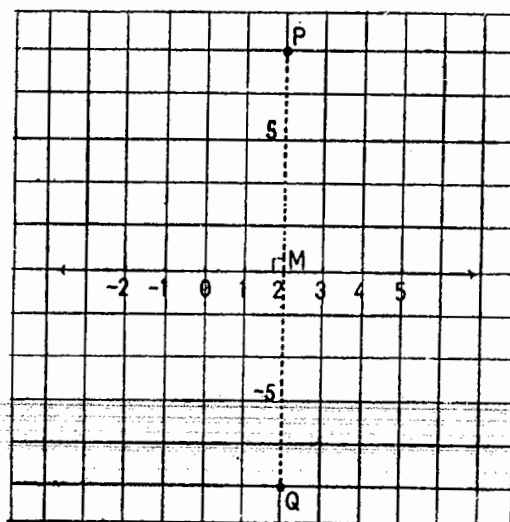


Fig. 2.9

A point on the number-line is determined by its co-ordinate. Let us now consider as to how to locate the point P on the plane. Let the perpendicular drawn from P on the number-line intersect it in M. Let the co-ordinate of M be 2. Let the length of the segment PM be 5. Then we can say,

‘The point P is at a distance of 5 units from the point on the number-line whose co-ordinate is 2.’ Is this description sufficient to determine the point P?

There is an infinite number of points which are 5 units away from the point M on the number-line in the given plane. If we modify the above description of P as ‘5 units away from M as well as from the number-line’, still we won’t be able to determine the position of P, as there are two points in the plane which are 5 units away from the number-line. The point Q in Fig. 2.9 is also at a distance of 5 units from the number-line.

To overcome this ambiguity we must find some way out. Instead of taking a single number-line, we take two number-lines in the plane which intersect at right angles in their zero-points. The same co-ordinate system is chosen on both the lines. Let us see how this would help us. On the horizontal number-line we establish the co-ordinate system as usual, i.e. points to the right of zero-point denote positive numbers and points to the left denote negative ones. On the vertical number-line,

we shall say, the points to the upward side of the zero-point will denote positive numbers and to the lower side, the negative numbers. For convenience, let us call the horizontal line, the X-line and the vertical line, the Y-line.

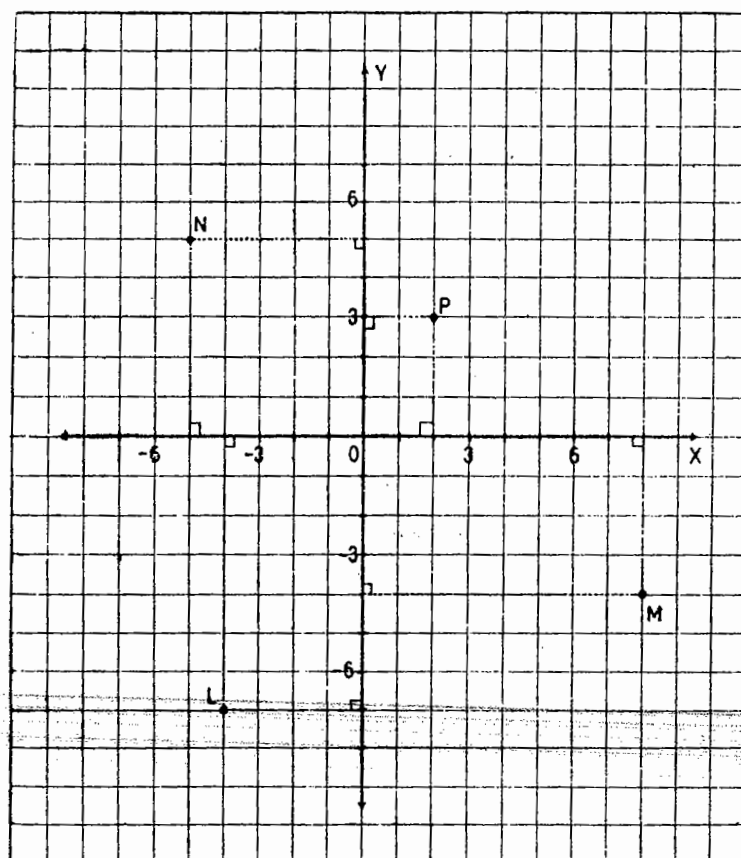


Fig. 2-10

Note from the adjoining figure that the distance of P from the Y-line is 2 while the distance of P from the X-line is 3. The point P can, therefore, be described by the ordered pair (2, 3). In this ordered pair, the first component is the distance of the point P from the Y-line and the second component is the distance of P from the X-line. Observing this convention we can describe the point N, by the ordered pair $(-5, 5)$ as it is to the left of the Y-line and to the upper side of the X-line. The

point L can be associated with the ordered pair $(-4, -7)$ as it is to the left of the Y-line and to the lower side of the X-line. Similarly, the ordered pair $(8, -4)$ represents the point M.

We shall henceforth call the reference lines X-line and Y-line as the **X-axis** and the **Y-axis** respectively. The distance of a point measured along a line parallel to the X-axis (i.e. its distance from the Y-axis) is called the **x-co-ordinate** of the point, and the distance of the point from the X-axis (measured along a line parallel to the Y-axis) is called the **y-co-ordinate** of the point. The x-co-ordinate of a point is also called the **abscissa** and the y-co-ordinate is known as the **ordinate**.

Thus, the point P in Fig. 2-10 is represented by the ordered pair (2, 3) which means that the x-co-ordinate of P is 2 and the y-co-ordinate is 3. Similarly, the co-ordinates of the point M are 8 and -4 . Any point in a plane has, with reference to the given co-ordinate system, two co-ordinates and they are denoted by an ordered pair in which the first number is the x-co-ordinate and the second one the y-co-ordinate of the point. With this convention the co-ordinates of the point N are represented by the ordered pair $(-5, 5)$ while those of L are given by the ordered pair $(-4, -7)$.

From the above discussion it is clear that the points in a plane and the ordered pairs of real numbers with reference to two intersecting number-lines, on which co-ordinate systems are established, are in one-one correspondence with each other. In this way, we are able to establish a co-ordinate system for the points in a plane.

Just as the points on a number-line and the corresponding numbers are known as 'the graphs' and 'the co-ordinates' respectively, in the same way, the points in a plane and the corresponding ordered pairs of real numbers are known as 'the graphs' and 'the co-ordinates' respectively. The point L is the graph of the ordered pair $(-4, -7)$ and the ordered pair $(-4, -7)$ gives the co-ordinates of the point L.

§ 2.5 Quadrants and Co-ordinates :

The four parts in which the plane is separated by the X-axis and the Y-axis (or the co-ordinate axes as they are called), are known as **quadrants**. Beginning from the right upper quadrant and going in the anti-clockwise direction, the quadrants are named as 'the first' (I), 'the second' (II), 'the third' (III) and 'the fourth' (IV) quadrants respectively. (See Fig. 2-11.)

The first quadrant lies above the X-axis and to the right of the Y-axis. Hence, both the co-ordinates of the points in that quadrant are positive. The third quadrant lies to the left of the Y-axis and below the X-axis. So both the co-ordinates of the points in the third quadrant are negative. The x-co-ordinate of a point in the second quadrant is negative while its y-co-ordinate is positive. The reverse is the case in respect of the points in the fourth quadrant, i.e. the x-co-ordinate

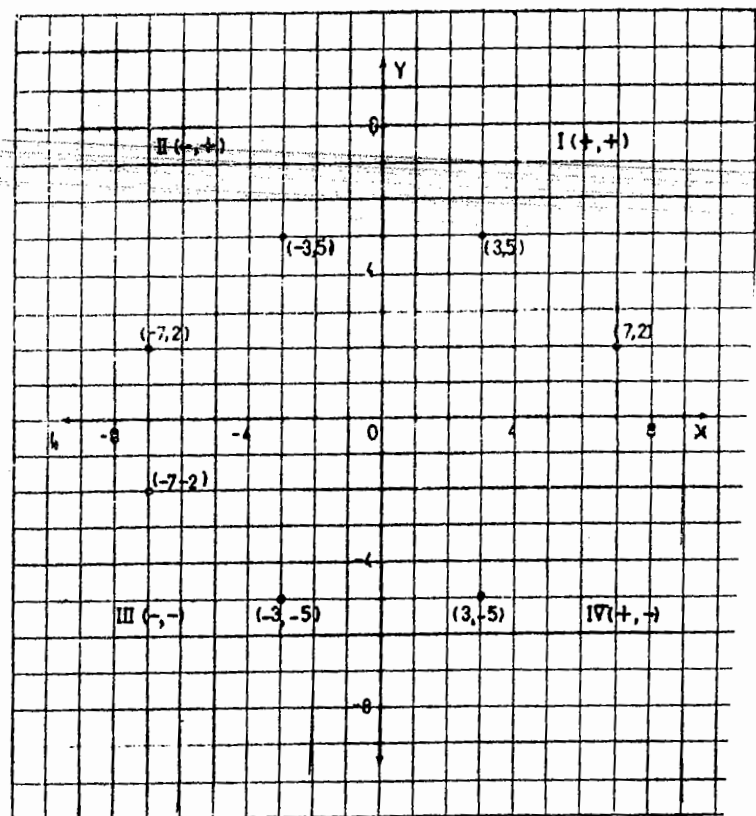


Fig. 2-11

of a point in the fourth quadrant is positive while its y-co-ordinate is negative.

According to the principle of separation of a plane, the points on the axes not lie in any of the quadrants. Nevertheless, they are the points of the plane. What co-ordinates are associated with these points on the axes? If we consider a point

the X-axis, it is evident that its distance from the Y-axis is its x -co-ordinate. The y -co-ordinate of such a point is its distance from the X-axis, which is naturally zero. Hence,

The y -co-ordinate of a point on the X-axis is zero.

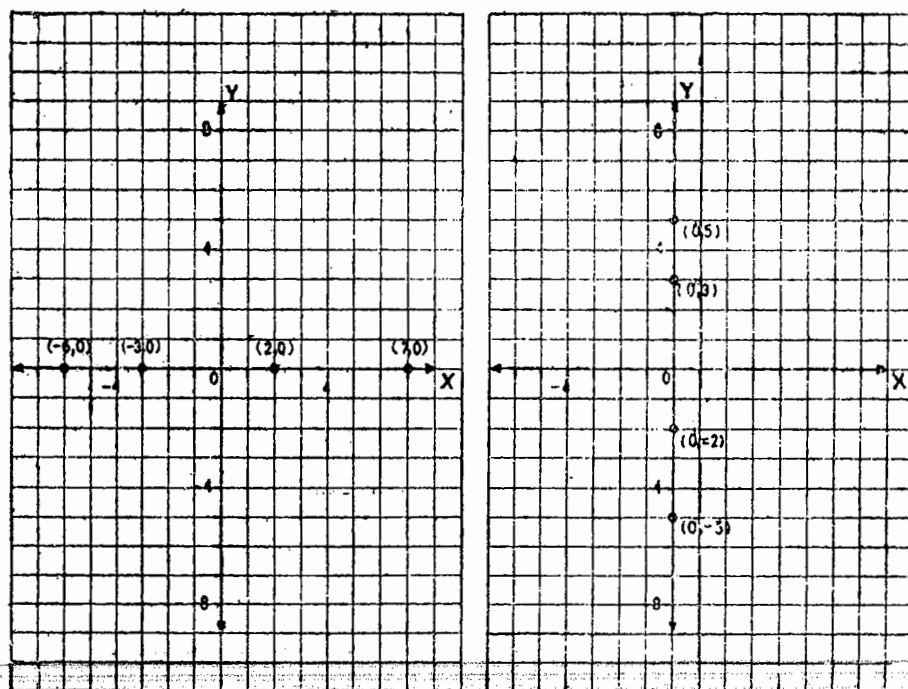


Fig. 2-12

What is the x -co-ordinate of a point on the Y-axis? The distance of a point on the Y-axis from this axis is naturally zero. Hence,

The x -co-ordinate of a point on the Y-axis is zero.

The co-ordinates of the point of intersection of the two axes are $(0, 0)$ and as we consider the distances of a point along the axes while determining the co-ordinates of the point, it is, the starting point. So, it is called the **origin**. The above discussion and Fig. 2-12 lead us to the following—

With reference to a given co-ordinate system, every point of a plane is associated with a unique ordered pair of real numbers, and conversely every ordered pair of real numbers represents a unique point of the plane with respect to the chosen co-ordinate axes.

EXERCISE 2-3

1. State the quadrant in which or on which axis the point represented by each of the following ordered pairs lies. In your graph-note-book establish a suitable co-ordinate system and mark the points represented by the ordered pairs and write their co-ordinates near them.

$(3, 4)$, $(-5, 0)$, $(5, -6)$, $(-2, -3)$, $(4, 2\frac{1}{2})$, $(2\frac{1}{2}, -4\frac{3}{4})$,
 $(6, -1\frac{1}{2})$, $(-1\frac{1}{2}, -6)$, $(-3\frac{1}{2}, 4\cdot5)$, $(2\cdot5, -3\cdot5)$, $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{3}, 0)$,
 $(0, -5)$, $(2\cdot5, 0)$, $(0, -6\cdot5)$.

2. Match the following ordered pairs with the statements given below.

(5, 0), (0, 0), (6, 10), (0, 5);

(i) $x = 0, y = 0$; (ii) 5 units' distance on the X-axis;

(iii) $y = 10, x = 6$; (iv) $x = 0, y = 5$.

3. Write the co-ordinates of the points shown in the adjoining figure.

4. Complete the following statements so as to make them true.

(i) The _____ co-ordinate of any point on the X-axis is zero.

(ii) The ordered pair (0, 0) represents the co-ordinates of _____.

(iii) The x -co-ordinate of a point is 0 and the y -co-ordinate of the said point is 6. Then the point must lie on the _____ axis.

(iv) ' $x = -5, y = 3$ ' represents a point in the _____ quadrant.

(v) Both the co-ordinates of a point in the third quadrant are _____.

(vi) The point $(7, -4\frac{1}{2})$ is in the _____ quadrant.

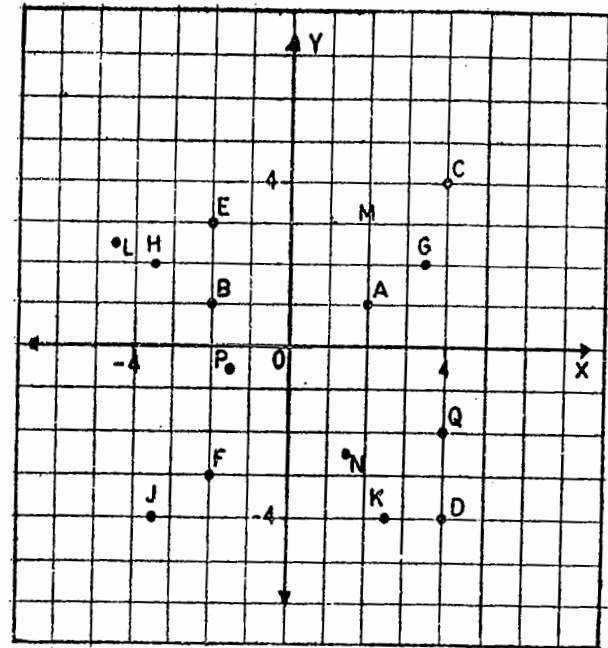


Fig. 2.13

§ 2.6 Lines Parallel to the Axes :

Observe the co-ordinates of the points A, B, C, D on the line AB in Fig. 2.1 on page 20. A is associated with the ordered pair (6, 9), B with (6, -8), C with (6, 7), and D with (6, -3). (1) Any point on the line AB has its x -co-ordinate 6, and (2) any point having its x -co-ordinate 6 must lie on the line AB. We, therefore, say that $x = 6$ is the equation of the line AB. The line AB is parallel to the Y-axis at a distance of 6 units from it.

Similarly, (1) if a point has its x -co-ordinate -3 , then it must lie on the line PS and (2) if a point lies on the line PS, its x -co-ordinate must be -3 . Hence, the equation of the line PS is $x = -3$. You will observe that the line PS is parallel to the Y-axis at a distance of -3 units from it.

All the points on a line parallel to the Y-axis have the same x -co-ordinate. Therefore, the equation of a line parallel to the Y-axis is of the form

$$x = a, (a \in \mathbb{R}).$$

If ' a ' is positive, the line is to the right of the Y-axis, and if ' a ' is negative it lies to the left of the Y-axis. What line would the equation $x = 0$ represent ?

' $x = 0$ ' is the equation of the Y-axis.

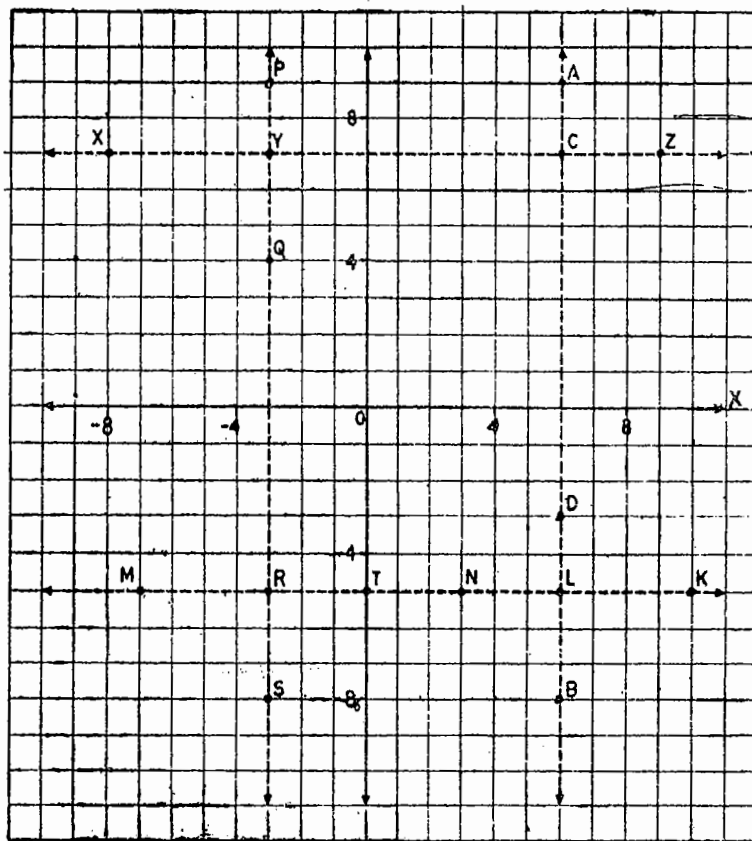


Fig. 2.14

Observe the co-ordinates of the points on the line XZ in the above figure. The y -co-ordinate of every point on the line XZ is 7. Also, any point having its y -co-ordinate 7, is on the line XZ. Hence, the equation of the line XZ is $y = 7$. Similarly, the equation of the line MK is $y = -5$. Both these lines are parallel to the X-axis.

All the points on a line parallel to the X-axis have the same y -co-ordinate. Therefore the equation of a line parallel to the X-axis is of the form—

$$y = b, (b \in \mathbb{R})$$

If ' b ' is positive, the line is above the X-axis, and when ' b ' is negative, it is below the X-axis. When ' b ' is zero, we have,

$$y = 0, \text{ the equation of the X-axis, itself.}$$

Understand, that the y -co-ordinate of any point on the X-axis is always zero, while the x -co-ordinate of any point on the Y-axis is always zero.

Now consider the co-ordinates of the point C in Fig. 2.14. C lies on the line AB, i.e. it is on the line whose equation is $x = 6$; it also lies on the line XZ whose equation is $y = 7$. This means that C is the point of intersection of the lines represented by the equations $x = 6$ and $y = 7$. Hence, the co-ordinates of point C are (6, 7).

The point R is the point of intersection of the lines PS and MK whose equations are $x = -3$ and $y = -5$ respectively. Therefore, the co-ordinates of the point R are (-3, -5).

EXERCISE 2.4

1. Refer to fig. 2.14 and state the co-ordinates of the points Q, M, L, Y and T. Write the equations of the two lines whose intersection is each of the above two points.
2. Complete the following statements to make them true.
 - (i) The equation of the line at a distance of -3 units from the X-axis is _____.
 - (ii) The equation of a line parallel to the Y-axis is _____.
 - (iii) The lines $x = -2$, $y = 0$ intersect at the point $(-, -)$.
 - (iv) The point $(5, -8)$ is on the line parallel to the X-axis having the equation _____.

§ 2.7 Some Graphs :

We have seen how points in a plane can be represented by ordered pairs of numbers. Plotting particular sets of ordered pairs we can get some idea about the graphs of certain familiar geometrical figures as well as of some interesting curves. Let us consider some of these. Figure 2.15 represents the graph of ordered pairs some of which are given in the following table.

x	0	1	-1	2	-2	3	-3	4	-4	5
y	1	0	2	1	3	2	4	3	5	4

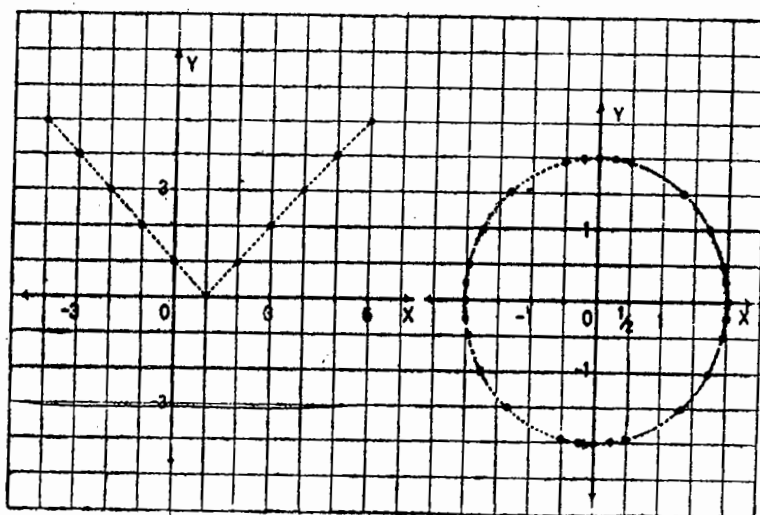


Fig. 2.15

Fig. 2.16

Fig. 2-16 shows points which when joined by a continuous curve give us a familiar curve. The table of ordered pairs representing the points plotted in fig. 2-16 is given below.

x	0	0	1	1	-1	-1	2	-2	1.5	-1.5
y	2	-2	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	0	0	1.3	1.3
x	1.75	-1.75	1.75	-1.75	0.5	0.5	-0.5	-0.5		
y	0.95	0.95	-0.95	-0.95	1.9	-1.9	1.9	-1.9		

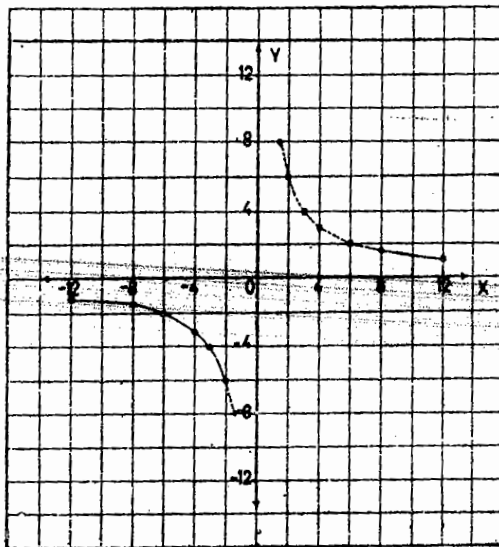


Fig. 2-17

An interesting figure is obtained when the points given by the following ordered pairs are joined smoothly. See fig. 2-18.

(0, 8), (0, -8), (1, 5), (1, -5), (2, 3.5), (2, -3.5), (3, 2.5), (3, -2.5), (4, 1.5), (4, -1.5), (5, 1), (5, -1), (6, 0.5), (6, -0.5), (8, 0), (-1, 5), (-1, -5), (-2, 3.5), (-2, -3.5), (-3, 2.5), (-3, -2.5), (-4, 1.5), (-4, -1.5), (-5, 1), (-5, -1), (-6, 0.5), (-6, -0.5).

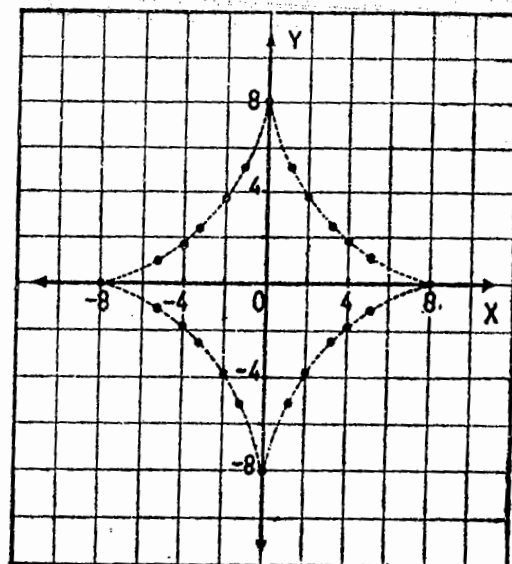


Fig. 2-18

Let us now be a bit rosy by plotting a number of points given below.

(0, 0), (1.5, -1.2), (4.5, -2.5), (8, -2.3), (10, 0), (8, 2.3), (4.5, 2.5), (1.5, 1.2), (-1.2, -1.5), (-2.5, -4.5), (-2.3, -8), (0, -10), (2.3, -8), (2.5, -4.5), (1.2, -1.5), (-1.5, 1.2), (-4.5, 2.5), (-8, 2.3), (-10, 0), (-8, -2.3), (-4.5, -2.5), (-1.5, 1.2), (1.2, 1.5), (2.5, 4.5), (2.3, 8), (0, 10), (-2.3, 8), (-2.5, 4.5), (-1.2, 1.5). (See Fig. 2-19 on page 23).

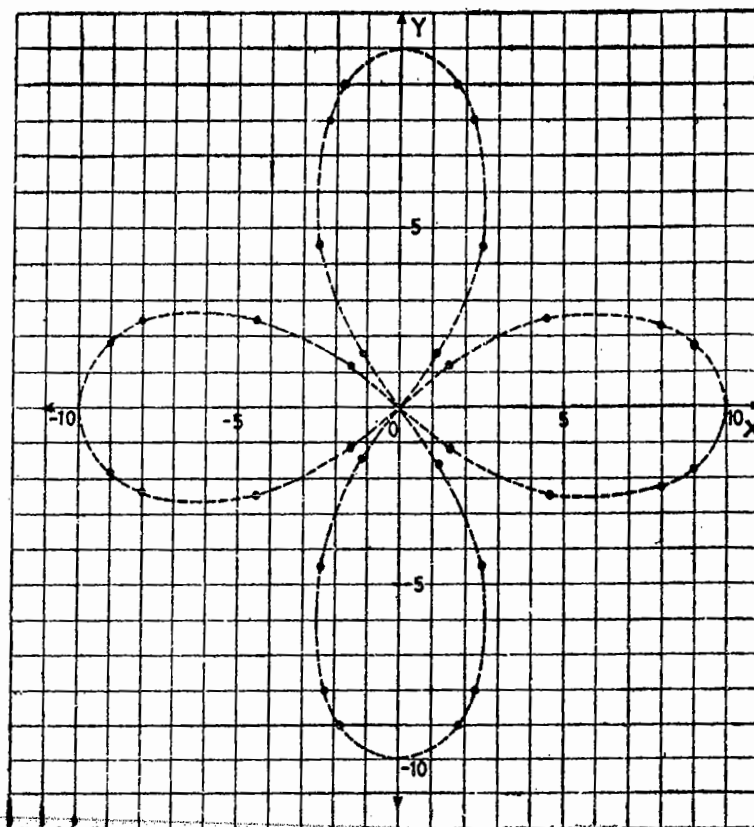


Fig. 2-19

EXERCISE 2-5

Plot the sets of points given by the following sets of ordered pairs separately. Take care to choose your units so that all the ordered pairs in each set are plotted. You will get amusing shapes of curve when you join the points by smooth curves.

1. $(1, 16), (2, 8), (4, 4), (6, 2.6), (8, 2), (16, 1), (-1, -16), (-2, -4), (-4, -4), (-6, -2.6), (-8, -2), (-16, -1)$.
2. $(1, 1), (-1, 1), (1.2, 0.7), (-1.2, 0.7), (1.5, 0.45), (-1.5, 0.4), (2, 0.25), (-2, 0.25), (3, 0.1), (-3, 0.1)$.
3. $(0, 0), (1, 1), (-1, -1), (2, 8), (-2, 8), (3, 27), (-3, -27), (4, 64), (-4, -64)$. [Choose a very small unit on the Y-axis.]
4. $(0, 0), (1, 0.5), (1, -0.5), (2, \sqrt{2}), (2, -\sqrt{2}), (3, 2.6), (3, -2.6), (4, 4), (4, -4), (5, 5.6), (5, -5.6)$. [$\sqrt{2} = 1.4$ approx.]

Chapter 3 : GRAPHS—1

§ 3.1 Open Sentences in Two Variables :

We know what is an 'open sentence'. The open sentences that we have considered so far contained only one variable. For instance, we considered open sentences like,

$$x + 3 < 5, \quad 2x - 6 = 9, \quad x \geq 0.$$

Some open sentences may contain two variables. For example,

$$x + y = 3, \quad 2x = 3y - 5, \quad x < y + 2, \quad 2x - y \geq 12,$$

are all 'open sentences in two variables'.

We shall study open sentences of the following two types—(i) equations and (ii) inequations, in two variables.

' $2x - 3 = 9$ ' is an equation in one variable, while ' $x + 3 < 5$ ' and ' $x \geq 0$ ' are inequations in one variable. Similarly, ' $x + y = 3$ ' and ' $2x = 3y - 5$ ' are examples of equations in two variables and ' $x < y + 2$ ' and ' $2x - y \geq 12$ ' are examples of inequations in two variables.

§ 3.2 Open Sentences and Ordered Pairs :

In the open sentence $x + y = 12$, if the value of x be 5, then the only value of y that will make the resulting open sentence true is 7. If we give different values to x , we shall have different corresponding values of y .

The table below contains some corresponding pairs of values of x and y that make the above open sentence true.

x	5	-5	$6\frac{1}{2}$	9.75	$6 + \sqrt{2}$	0
y	7	17	$5\frac{1}{2}$	2.25	$6 - \sqrt{2}$	12

There is an infinite number of such pairs of values of x and y which will make the above open sentence true. If it is decided to state the pairs of the corresponding values of x and y in such a way that the first element of the pair will always denote the value of x and the second element will denote the corresponding value of y , then the above pairs of values can be written as ordered pairs as follows.

$$(5, 7), (-5, 17), (6\frac{1}{2}, 5\frac{1}{2}), (9.75, 2.25), (6 + \sqrt{2}, 6 - \sqrt{2}), (0, 12).$$

With the above convention, the corresponding values of the variables of an open sentence in two variables form a set of ordered pairs. We know that ordered pairs of real numbers represent points in a plane. Hence, the graph of an open sentence in two variables must be a set of points in a plane.

§ 3.3 Graph of $y = x$:

Let us consider the simplest open sentence in two variables ' $y = x$ '. Make a table in your note-book as shown below and complete it by filling in the blanks.

$$y = x$$

x	2	5	-3	-6		0	3	4		
$y = x$	2	5	-3		-8				-2	-4

The above table contains the following ordered pairs.

(2, 2), (5, 5), (-3, -3), (-6, -6), (-8, -8), (0, 0), (3, 3), (4, 4), (-2, -2), (-4, -4).

Establish a co-ordinate system on a graph paper and plot the points representing the above ordered pairs. The plotted points seem to lie on a straight line, as in fig. 3.1.

There is an infinite number of points on the line other than those plotted by you. Consider the ordered pair of the co-ordinates of any point on the line, other than those plotted by you. Suppose, we choose, (6, 6) representing the co-ordinates of a point on the line. This ordered pair means $x = 6$, $y = 6$. These values of x and y do satisfy the open sentence ' $y = x$ '. Further, you will notice that the point represented by the ordered pair

(-5, -5) is on the line you have drawn. The values of x and y in this ordered pair also make the open sentence ' $y = x$ ' to be true. The point having co-ordinates (2.5, 2.5) also lies on the line and its co-ordinates also satisfy the open sentence ' $y = x$ '. In conclusion, we may say,

(1) that the co-ordinates of any point on the line make the open sentence ' $y = x$ ' to be true, and

(2) that the points whose co-ordinates are obtained from the given open sentence ' $y = x$ ' lie on that line.

Hence, the line is called the **graph of the open sentence ' $y = x$ '**. So, the line is ' $y = x$ ' (as shown in the figure 3.1).

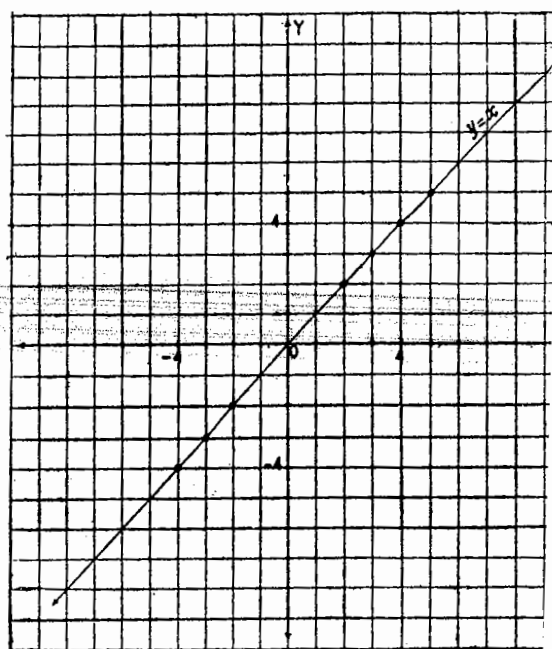


Fig. 3.1

§ 3.4 Graph of ' $y = -x$ ' :

Copy and complete the following table for the ordered pairs obtained from the open sentence ' $y = -x$ '.

$y = -x$										
x	3	6	-2	0			4	2.5	-2.5	
$y = -x$	-3	-6	2		-5	5				4.5

Plot the points represented by the ordered pairs in the above table, on the same co-ordinate system used by you for the graph of ' $y = x$ '.

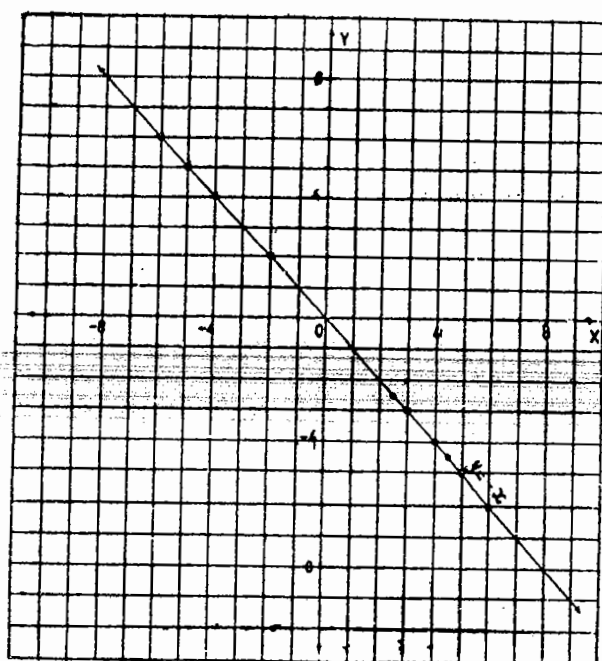


Fig. 3.2

Do you feel that the points plotted by you lie on a straight line? If so, join them with a straight line. Choose some points on the line, not plotted by you and see whether their co-ordinates satisfy the open sentence ' $y = -x$ '. The graph of $y = -x$ is also a straight line. Name the line on your graph-paper. (See Fig. 3.2.)

Comparing the two graphs of ' $y = x$ ' and ' $y = -x$ ' you will notice that—

- (1) Both the graphs pass through the origin (0, 0).
- (2) Both the graphs are straight lines.
- (3) The graph of $y = x$, other than the point (0, 0), lies in the first and the third quadrants. The point (0, 0) also lies on the graph; while the graph of $y = -x$ lies in the second and the fourth quadrants and passes through the origin.
- (4) Both the graphs bisect the vertically opposite angles between the axes.

§ 3.5 Graph of ' $y = x + h$ ' :

We have seen how the open sentence ' $y = x$ ' is represented by a line through the origin. Now let us draw the graph of ' $y = x + h$ ', where $h \in \mathbb{R}$, $h \neq 0$.

If the given equation is $y = x$, then both the co-ordinates of a point on its graph are equal. Instead, if we have the equation $y = x + h$, it means that the y -co-ordinate of a point on its graph exceeds its x -co-ordinate by h , if h is positive, and is less than by $|h|$ if h is negative. Consequently, every point on the line $y = x$, if translated along the line parallel to the Y -axis ' h ' units above or below according as $h > 0$ or $h < 0$, it will be a point on the graph of $y = x + h$.

The important property of translation is that every point on a line or a ray or a segment or a curve moves the same distance in the direction of translation, hence the image of the line or the ray or the segment after translation is parallel to the original one. Hence, the graphs of open sentences of the form $y = x + h$ must be parallel to that of $y = x$.

In particular, let us see what happens to the point $(0, 0)$ on $y = x$, when the open sentence is $y = x + h$. The y -co-ordinate of the point will be increased or decreased by $|h|$ according as h is positive or negative. So, the point $(0, 0)$ is translated to the point $(0, h)$. So, the graph of $y = x + h$ cuts the Y -axis in $(0, h)$ and is parallel to the line which is the graph of $y = x$. The figure 3.3 shows how the points on $y = x$ are displaced. Some points on the line $y = x$ are shown as $\odot$.

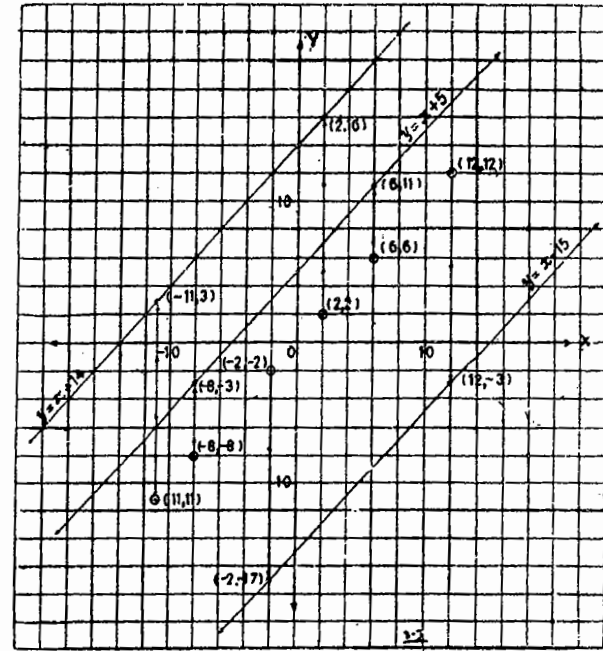


Fig. 3.3

If ' h ' is positive, the graph of $y = x + h$ cuts the Y -axis in a point on the positive side of the Y -axis and if ' h ' is negative it cuts the Y -axis in a point on its negative side.

EXERCISE 3.1

Prepare and complete tables as shown below for obtaining co-ordinates of some points on the graphs of the equations shown above each table.

$$y = x$$

x	0	-5	-10	3	8		
$y = x$	0					-12	12

$$y = x + 6$$

x	0	-5	-10	3	8	-12	12
$y = x + 6$							

$$y = x - 8$$

x	0	-5	-10	3	8	-12	12
$y = x - 8$							

Plot the points in each of the tables on the same co-ordinate system and draw the lines passing through each of the sets of points plotted. Show how the points $(-5, -5)$ $(8, 8)$ on the line $y = x$ are displaced for the graph of $y = x + 6$. Similarly, show how the points $(-10, -10)$, $(3, 3)$ on $y = x$ are displaced on the line $y = x - 8$.

§ 3.6 Slope of a Segment :

Along the railway lines you must have seen signs like $\begin{array}{|c|} \hline \uparrow \\ \hline 500 \\ \hline \end{array}$ or $\begin{array}{|c|} \hline 200 \\ \hline \downarrow \\ \hline \end{array}$. The sign $\begin{array}{|c|} \hline \uparrow \\ \hline 500 \\ \hline \end{array}$ indicates that the track is raised above by 1 metre in a length of 500 metres. The sign $\begin{array}{|c|} \hline 200 \\ \hline \downarrow \\ \hline \end{array}$ indicates that there is a fall of 1 metre in the track per 200 metres.

The rise or the fall in the height shows the 'slope' of the track. A rise means that the slope is upwards, while a fall means that the slope is downwards. (See Fig. 3.4)

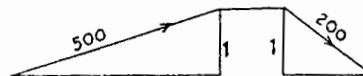


Fig. 3.4

In mathematics, however, the convention of stating the slopes of lines or segments is a bit different.

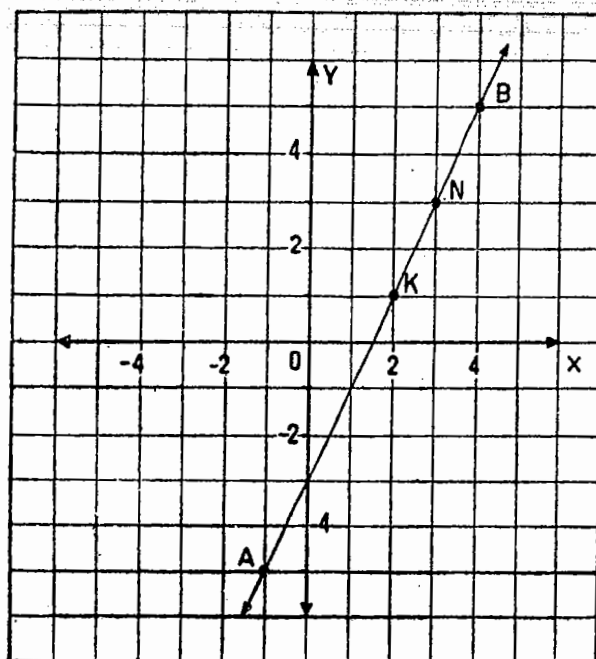


Fig. 3.5

Consider the segment NK of the line AB. The co-ordinates of K are $(2, 1)$ while those of N are $(3, 3)$. Take the difference between the y-co-ordinates of N and K and divide it by the difference in the x-co-ordinates of N and K.

$$\begin{aligned} \text{Thus, } \frac{\text{change in y-co-ordinates}}{\text{change in x-co-ordinates}} \\ = \frac{3 - 1}{3 - 2} = \frac{2}{1} = 2. \end{aligned}$$

This ratio of the difference in the y-co-ordinates to the difference in the x-co-ordinates is known as the slope of seg. NK.

Now let us find the slope of seg. AB of the line AB.

$$\begin{aligned} \text{Slope of seg. AB} &= \frac{(y\text{-co-ordinate of A}) - (y\text{-co-ordinate of B})}{(x\text{-co-ordinate of A}) - (x\text{-co-ordinate of B})} \\ &= \frac{(-5) - 5}{(-1) - 4} = \frac{-10}{-5} = 2. \end{aligned}$$

$$\begin{aligned}\text{Similarly, slope of seg. KB} &= \frac{(y\text{-co-ordinate of K}) - (y\text{-co-ordinate of B})}{(x\text{-co-ordinate of K}) - (x\text{-co-ordinate of B})} \\ &= \frac{1 - 5}{2 - 4} = \frac{-4}{-2} = 2.\end{aligned}$$

$$\begin{aligned}\text{The slope of seg. NA} &= \frac{\text{change in } y\text{-co-ordinates}}{\text{change in } x\text{-co-ordinates}} \\ &= \frac{3 - (-5)}{3 - (-1)} = \frac{8}{4} = 2.\end{aligned}$$

From the above discussion, we see that the slopes of any of the segments of a line are the same. Hence, the **slope of any line** is the **slope** of any of its segments.

Now let us consider the slope of the line PQ in fig. 3.6. The co-ordinates of the two points L and M of that line are $(-2, 2)$ and $(0, -1)$ respectively.

$$\begin{aligned}\text{So, the slope of seg. LM} \\ &= \frac{2 - (-1)}{(-2) - (0)} = \frac{3}{-2} = -\frac{3}{2}\end{aligned}$$

$$\begin{aligned}\text{The slope of seg. MN} &= \frac{(-1) - (-4)}{0 - 2} \\ &= \frac{3}{-2} = -\frac{3}{2}\end{aligned}$$

$$\text{So, the slope of the line PQ is } -\frac{3}{2}$$

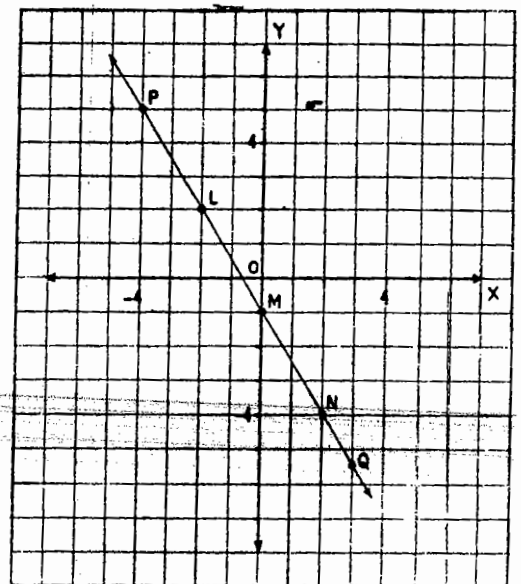


Fig. 3.6

What do we mean by the slope being negative ?

Look at the line AB in fig. 3.5. It makes an acute angle with the positive side of the X-axis. The line PQ in fig. 3.6 makes an obtuse angle with the positive side of the X-axis. These angles are to be reckoned from the positive side of the X-axis in the anti-clockwise rotation. The slope of a line making an acute angle with the positive side of X-axis is positive, while that of a line making an obtuse angle with the positive side of X-axis is negative.

§ 3.7 Graph of ' $y = mx$ ', ($m \in \mathbb{R}$):

The sets of ordered pairs below contain ordered pairs in some sequence. Copy them in your note-book and (i) Write out two more pairs in the sequence for each case, (ii) Then plot the points given by the ordered pairs in each of the sets and see if they lie on a straight line in each case. If so, draw those lines.

- (a) $\{(-2, -2), (-1, -1), (0, 0), (1, 1), (,), (,)\}$
- (b) $\{(-2, -4), (-1, -2), (0, 0), (1, 2), (,), (,)\}$
- (c) $\{(-6, -3), (-2, -1), (2, 1), (6, 3), (,), (,)\}$
- (d) $\{(-2, -3), (-1, -1.5), (1, 1.5), (2, 3), (,), (,)\}$

The y -co-ordinate in each of the ordered pairs in (a) above is equal to the x -co-ordinate in the pair. Therefore, the points represented by these ordered pairs must be on the line whose equation is $y = x$.

The y -co-ordinate in each of the pairs in (b) above, is twice the x -co-ordinate. That is, $y = 2x$. Hence, the line on which the points are represented by the ordered pairs in (b) should be named as $y = 2x$.

Find out the relation between the y -co-ordinate and the x -co-ordinate in the ordered pairs in the sets (c) and (d) above, and name the lines properly.

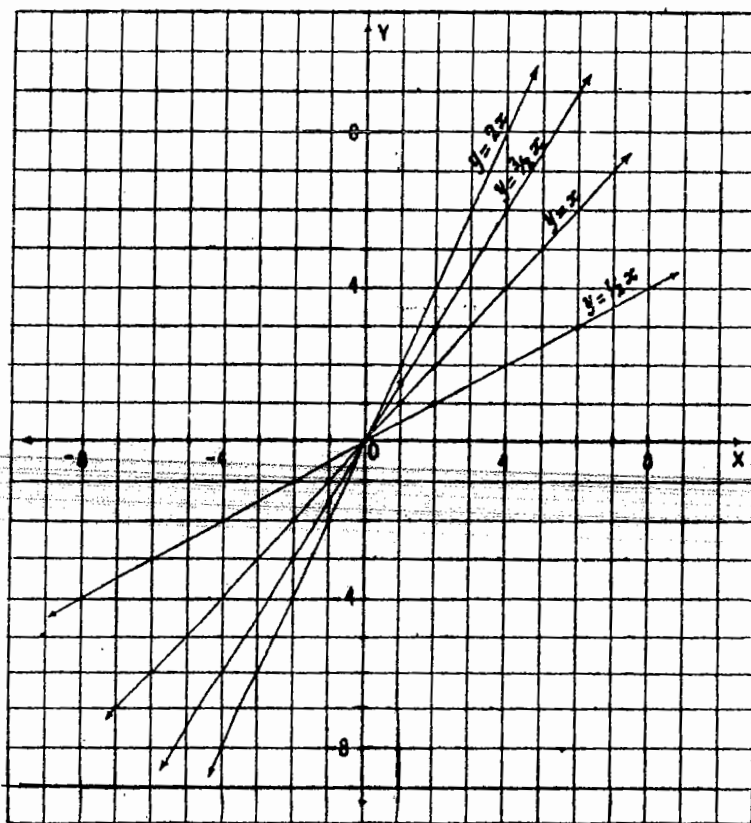


Fig. 3.7

Observe the lines and the respective open sentences, which they represent. The angle made by each line with the ray OX and the coefficient of x in the open sentence have a close relation. As the coefficient of x increases, the measure of the angle also increases. All the open sentences representing the lines in the Fig. 3.7 are of the type $y = mx$ where m is a positive real number. As the value of m increases, the angle made by the line with

the positive side of the X -axis increases.

Let us now consider the slopes of the lines drawn in Fig. 3.7.

- (1) Choose any two points on the line $y = x$. Say, $(-2, -2)$ and $(5, 5)$. Then

$$\text{the slope of the line} = \frac{(-2) - 5}{(-2) - 5} = \frac{-7}{-7} = 1.$$

- (2) Let us choose the points $(-2, -4)$ and $(3, 6)$ on the line $y = 2x$. The

$$\text{slope of the line} = \frac{(-4) - (6)}{(-2) - (3)} = \frac{-10}{-5} = 2.$$

- (3) If we choose the points $(6, 3)$ and $(-2, -1)$ on the line $y = \frac{1}{2}x$, the slope

$$\text{is seen to be } \frac{3 - (-1)}{6 - (-2)} = \frac{4}{8} = \frac{1}{2}.$$

(4) Taking the two points $(-1, -1.5)$ and $(4, 6)$, on the line $y = \frac{3}{2}x$, the

$$\text{slope is seen to be } \frac{(-1.5) - 6}{(-1) - 4} = \frac{-7.5}{-5} = \frac{3}{2}.$$

Let us tabulate the above results.

Equation of the line	Slope of the line
$y = x$	1
$y = 2x$	2
$y = \frac{1}{2}x$	$\frac{1}{2}$
$y = \frac{3}{2}x$	$\frac{3}{2}$

Do you see any connection between the number denoting the slope and the corresponding equation? The coefficient of x in the equation is equal to the slope of the line. You will further notice that as the slope increases from $\frac{1}{2}$ to 1, $\frac{3}{2}$, 2, respectively, the angle made by the line with the ray OX also increases.

In general, we can say,

$y = mx$, ($m \in \mathbb{R}$ and m is positive), represents lines through the origin.

' m ' represents the slope of the respective lines. As ' m ' increases, the angle which the line makes with ray OX increases in magnitude.

Now complete the following tables with reference to the equations given above each of them, and draw the graphs on one and the same co-ordinate system.

$$y = -x$$

x	4	-5	0	-3		6		-4
$y = -x$	-4	5			-3		6	

$$y = -\frac{1}{2}x$$

x	0	4	6		-8	-6	-2	
$y = -\frac{1}{2}x$			-3	2	4		1	-4

$$y = -\frac{3}{2}x$$

x	2	4	6				5	
$y = -\frac{3}{2}x$	-3			6	9	0	-7.5	4.5

$$y = -2x$$

x	2	-2	-4	1.5				0
$y = -2x$	-4				3	-8	-6	

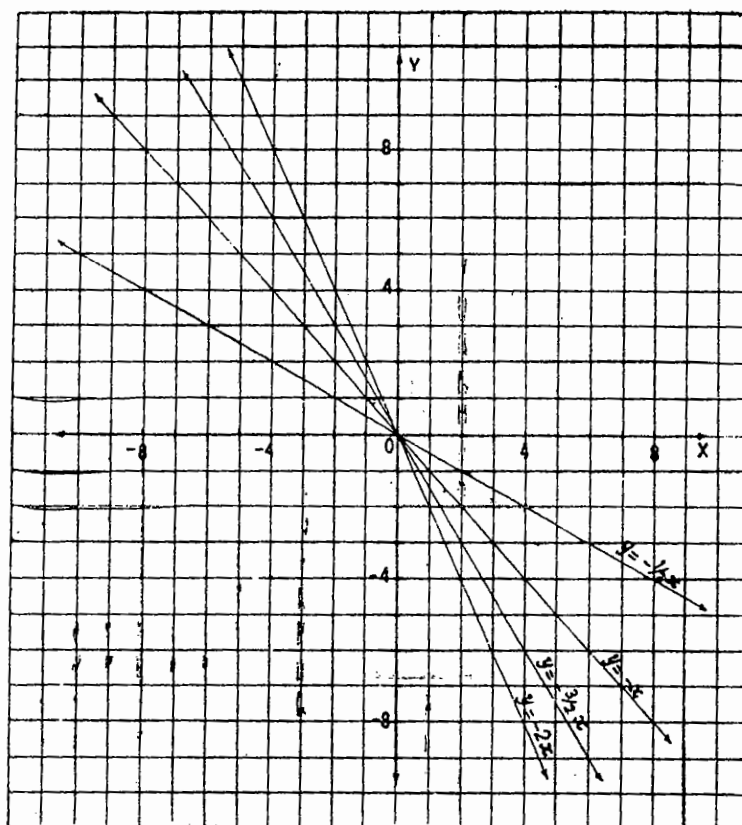


Fig. 3.8

Now choose any two points on each of the lines and find their slopes.

The graphs of the equations given on page No. 31 are shown in Fig. 3.8.

You will observe that these equations are also of the form $y = mx$, but that the values of ' m ' are negative. The same relationship as observed above, exists between the slopes and the coefficients of x in the equations. Note that the slopes are negative and the angles made by the

lines with the ray OX in the anti-clockwise sense are obtuse. You may further observe that $-2 < -\frac{3}{2} < -1 < -\frac{1}{2}$. As the slope increases through negative values from -2 to $-\frac{1}{2}$, the obtuse angle made by the lines also increases.

To summarise, observe the following facts regarding the graphs of the open sentences of the form $y = mx$, ($m \in \mathbb{R}$) —

- (1) The graphs are straight lines;
- (2) The graphs pass through the origin, i.e. the point $(0, 0)$;
- (3) If ' m ' is positive the graph makes an acute angle with the ray OX. If m is negative the graph makes an obtuse angle with the ray OX.

EXERCISE 3.2

1. With reference to the figure 3.9 on page 33 complete the following statements.
 - (a) The line AB is the graph of $y = \square x$.
 - (b) The equation of the line PQ is $y = \square x$.
 - (c) The points $(2, 1)$, $(3, 1\frac{1}{2})$, $(4, 2)$, $(-6, -3)$ among others lie on the line——, whose equation is the open sentence——.
 - (d) The points $(2, \square)$, $(-4, \square)$, $(\square, -10)$ lie on the line RS whose equation is——.

(e) The point (4, 6) cannot lie on the line LK, as the relation between the co-ordinates of the given point is $y = \square x$, while the equation of the line LK is ———.

2. Write out the equations of the lines passing through the following sets of points :—

(a) { (5, 10), (3, 6), (14, 28), (501, 1002), (0.5, 1) }

(b) { (−10, 5), (6, −3), (28, −14), (−1002, 501), (1, −0.5) }

(c) { (8.6, 4.3), (−10, −5), (−27.2, −13.6), (0, 0), $(2\sqrt{2}, \sqrt{2})$ }

3. Order the following equations according to the increasing magnitude of the angles which their graphs make with the ray OX.

(a) $y = 7x$; (b) $y = -\frac{1}{2}x$; (c) $y = -3x$; (d) $y = \frac{3}{4}x$.

4. If you reflect the points (1, 2), (4, 8), (−3, −6), (−6, −12) in the X-axis, what will be the co-ordinates of their images? Are the given points on one and the same line? If so, write down the equation of the line. Are the images of these points on one and the same line? If so, write down the equation of the line of the images.
5. If the points given in Ex-4 above are reflected in the Y-axis, what would be their co-ordinates of their images?
6. Name the points given in Ex-4 above as A, B, C, D in that order and plot them on a co-ordinate system. Plot their images after reflection in the X-axis and name them as A', B', C', D', respectively. Draw the lines passing through both the sets of points and give their equations.
7. Draw the graphs of the following equations on the same co-ordinate system
(i) $y = -1\frac{1}{2}x$; (ii) $y = 2.5x$; (iii) $y = -\frac{5}{2}x$; (iv) $2y = 3x$.

§ 3.8 Graph of $y = mx + h$:

We have seen how the term 'h' in the equation $y = x + h$ or $y = -x +$ translates the graph of $y = x$ or $y = -x$ respectively. The effect of 'h' in the equation $y = mx + h$ is the same as in the above two cases. That is, the graph of $y = mx$ is displaced by 'h' in the direction of the Y-axis; then this new line is the line having $y = mx + h$ as its equation. If every point on the line $y = n$

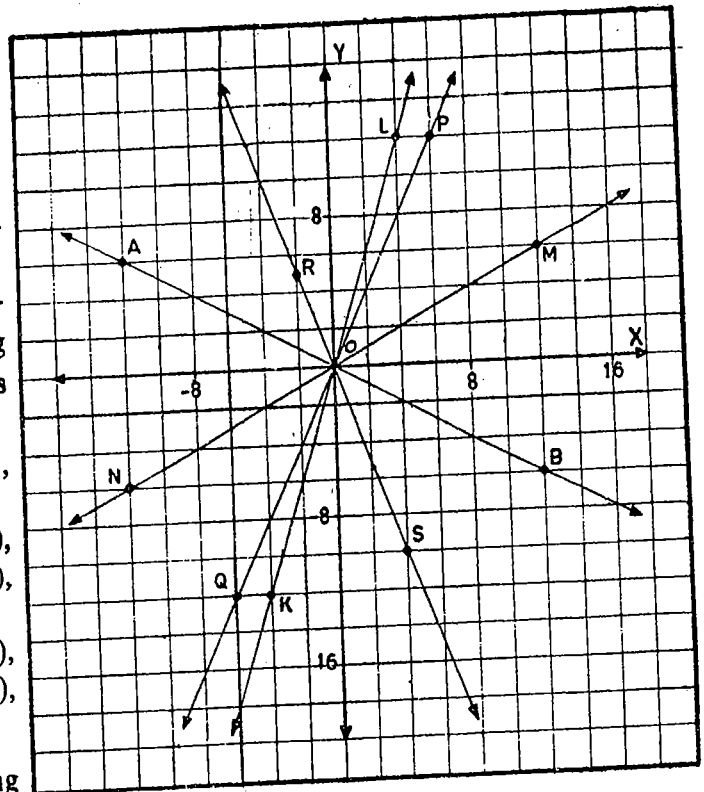


Fig. 3.9

is displaced by 'h' along a line parallel to the Y-axis, then its images thus obtained will lie on the line $y = mx + h$.

The point which lies on the line $y = 2x$ if translated 4 units upwards, will now lie on the line $y = 2x + 4$. Similarly, the point $(-3, -6)$ lying on the line $y = 2x$ if displaced 6 units downwards (i. e. if $h = -6$), will have co-ordinates $(-3, -12)$ and it will lie on the line $y = 2x - 6$.

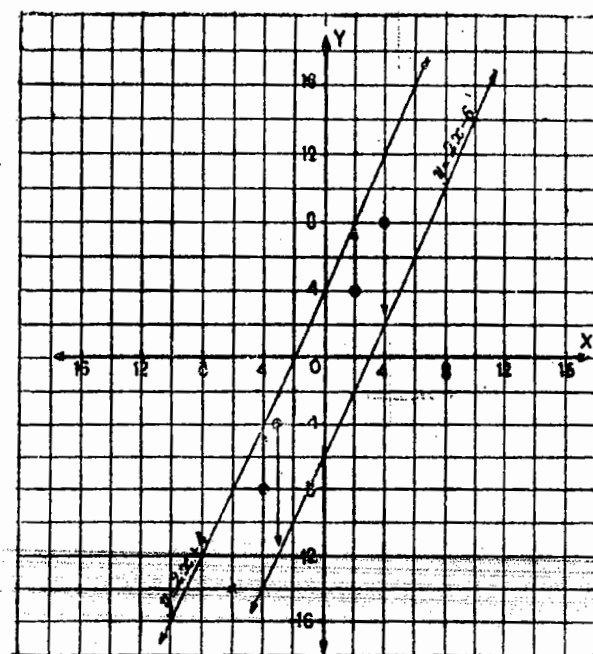


Fig. 3.10

Observe the displacements of points $(4, 8)$ and $(-4, -8)$ shown in the adjoining figure.

The point $(0, 0)$ which lies on the graph of the equation $y = 2x$ is translated to the point $(0, 4)$ so that its image $(0, 4)$ lies on the graph of $y = 2x + 4$. If it is translated to the point $(0, -6)$ after translation by 6 units downwards it lies on the line $y = 2x - 6$. In other words, the line $y = 2x + 4$ cuts the Y-axis in $(0, 4)$ and the line $y = 2x - 6$ cuts the Y-axis in $(0, -6)$.

We say that the line $y = 2x + 4$ makes an intercept 4 on the Y-axis,

and the line $y = 2x - 6$ makes an intercept (-6) on the Y-axis.

Now, plot the following points on a co-ordinate system, $(4, -6)$, $(-4, 6)$, $(8, -12)$, $(-8, 12)$. The equation of the line passing through these points is $y = -\frac{3}{2}x$. Displace each of the above points 5 units upwards. Then the images of those points will respectively be $(4, -1)$, $(-4, 11)$, $(8, -7)$, $(-8, 17)$ and these will lie on the line $y = -\frac{3}{2}x + 5$.

If the points given above be displaced 3 units downwards, what will be the co-ordinates of their images and what will be the equation of the line on which the images lie ?

§ 3.9 Intercepts on the Axes :

We have seen that the graph of $y = mx + h$ represents a line which cuts the Y-axis in the point $(0, h)$ and we say that the line makes an intercept 'h' on the Y-axis. The x-co-ordinate of every point on the Y-axis is zero. Hence, if we put $x = 0$ in a given equation, we get the value of y, which corresponds to the value of the intercept on the Y-axis.

Similarly, the y-co-ordinates of every point on the X-axis is zero. So, if we put $y = 0$ in the given equation the corresponding value of x satisfying the equation gives us the intercept on the X-axis made by the line representing the given equation.

Let $y = 2x - 3$ be the given equation. Here, $m = 2$, $h = -3$. If we put $x = 0$ in the equation, we get $y = -3$. Therefore, the y -intercept in this case is $(0, -3)$. If we put $y = 0$, we get $x = \frac{3}{2}$. Therefore, the x -intercept is $(\frac{3}{2}, 0)$. Thus $y = 2x - 3$ cuts the Y -axis in $(0, -3)$ and the X -axis in $(\frac{3}{2}, 0)$.

Verify that $(0, -3)$ and $(\frac{3}{2}, 0)$ satisfy the equation $y = 2x - 3$.

Hence, the one method for drawing the graph of an equation of the form $y = mx + h$ is to find the intercepts on the two axes and plot those points and draw the line passing through them. Sometimes, we may get very odd values for the intercepts, and it makes it impossible to plot these points and therefore the given line.

We know that a line is determined by two points. Hence, in order to be able to draw the graph of a given equation of a line, it would suffice to get the co-ordinates of two points satisfying the given equation.

Prepare and complete the tables as shown below with reference to the equation given above.

(i) $y = 2x + 8$

x	0		-8
$y = 2x + 8$		0	

(iii) $y = -3x + 6$

x	0		6
$y = -3x + 6$		0	

(ii) $y = -x - 10$

x	0		-
$y = -x - 10$		0	

(iv) $y = 3x - 7$

x	0	5	-
$y = 3x - 7$			

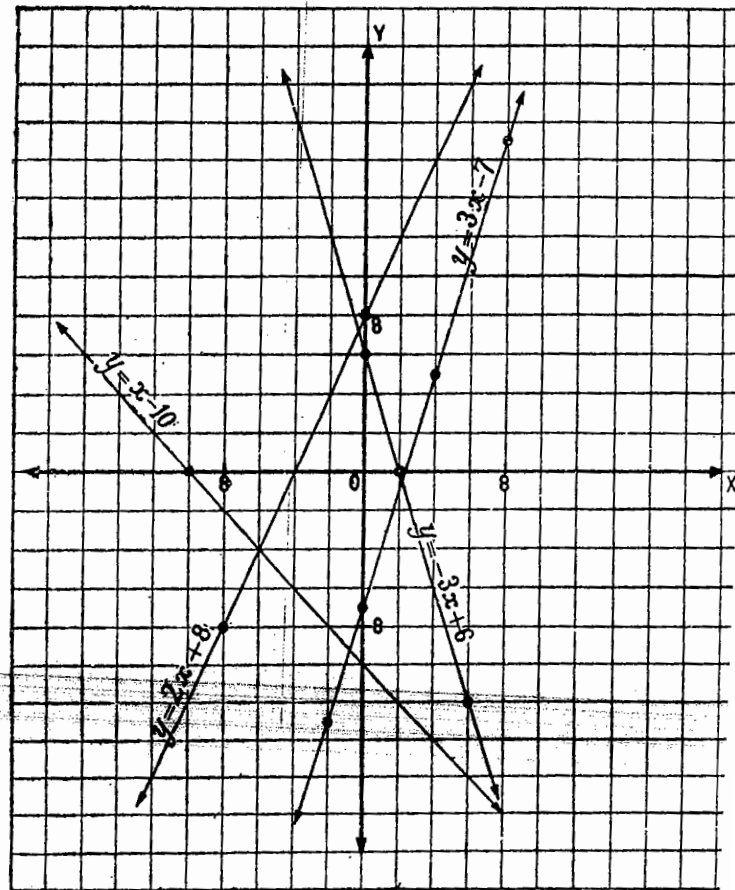


Fig. 3-11

For the fourth equation, the y -intercept is (-7) , but the value of the x -intercept is fractional. Hence, we choose some other convenient value for x .

Plot the points with the help of the above tables on one and the same co-ordinate system and draw the lines passing through them. Their graphs are shown in fig. 3.11 on page 35.

EXERCISE 3.3

- Plot the following points on a co-ordinate system.
 $(-1, -1), (1, 3), (2, 5)$. Do these points lie on a straight line? If so, draw that line.
 - What relation do you notice between the y -co-ordinates and the x -co-ordinates of the above points?
 - If $(3, \square)$ and $(\square, -5)$ are points on the line passing through the above points, what numbers should replace the $\square$ (boxes) in the two ordered pairs given above?
 - Draw the graph of $y = 2x$ on the same co-ordinate system.
 - Write down the equation of the first line you have drawn in this question.
- In which point does the graph of $y = 2x + h$, ($h \in \mathbb{R}$), cut the Y -axis? In which point does it cut the X -axis?
- Draw the graphs of the following equations on the same co-ordinate system :
 (i) $y = 2x$; (ii) $y = 2x + 4$; (iii) $y = 2x - 3$.
 Do you notice any special thing about these graphs? Can you give any reason supporting the observation you have made?
- Plot the points $(2, 1), (4, -3), (-3, 11)$. Join them. Do they lie on the same straight line? Draw the graph of $y + 2x = 0$ on the same co-ordinate system. Write down the equation of the line passing through the three given points. What is the intercept made by the line on the Y -axis?

§ 3.10 Linear Equation :

Consider the equation : $2x - 3y + 6 = 0$.

We subtract $(2x + 6)$ from both sides and obtain the equivalent equation
 $-3y = -2x - 6$.

Dividing both sides by (-3) , we obtain another equivalent equation
 $y = \frac{2}{3}x + 2$.

This equation is now in the familiar form $y = mx + h$, ' m ' being $\frac{2}{3}$ and ' h ' being 2 in this case.

The given equation was in the form $Ax + By + C = 0$, wherein $A = 2$, $B = -3$, $C = 6$. The said equation was transformed into an equivalent equation of the form $y = mx + h$. You will observe that—

If $Ax + By + C = 0$, ($B \neq 0$), be transformed into $y = mx + h$, then

$$m = \frac{-A}{B} \text{ and } h = \frac{-C}{B}.$$

Let us verify this fact in case of the above given equation. In the given equation. $A = 2$, $B = -3$, $C = 6$.

$$\text{Hence, } m = \frac{-A}{B} = \frac{-2}{-3} = \frac{2}{3}, \quad h = \frac{-C}{B} = \frac{-6}{-3} = 2,$$

which tallies with the equation $y = \frac{2}{3}x + 2$.

$Ax + By + C = 0$, ($B \neq 0$), is called the **general equation of the first degree in two variables**. We know that $y = mx + h$ is the **slope-intercept form** of an equation to a line. We can transform any equation given in the general form into an equivalent equation in the 'slope-intercept' form provided the coefficient of y is other than zero.

Example 1 : Write the equation $2x + 3y + 8 = 0$ in the 'slope-intercept' form and state the values of the slope and the intercept on the Y-axis made by its graph.

$$2x + 3y + 8 = 0.$$

Subtracting $(2x + 8)$ from both sides, we get

$$3y = -2x - 8.$$

Dividing both sides by 3, we get,

$$y = -\frac{2}{3}x - \frac{8}{3},$$

which is in the form $y = mx + h$, giving $m = -\frac{2}{3}$, $h = -\frac{8}{3}$.

Hence, the slope of the graph is $-\frac{2}{3}$ and the intercept on the Y-axis is $-\frac{8}{3}$.

Example 2 : Transform the equation $2y - 5x - 6 = 0$ in the form $y = mx + h$ and state the slope of its graph and the intercepts on both the axes.

$$2y - 5x - 6 = 0$$

Adding $(5x + 6)$ to both sides, we get,

$$2y = 5x + 6.$$

Multiplying both sides by $\frac{1}{2}$, we get,

$$y = \frac{5}{2}x + 3.$$

$$\therefore m = \frac{5}{2}, h = 3.$$

$\therefore$ The slope of the graph is $\frac{5}{2}$ and the intercept on the Y-axis is 3.

To find the intercept on the X-axis, we put 0 for y in $y = \frac{5}{2}x + 3$ and get $0 = \frac{5}{2}x + 3$

$$\therefore x = -\frac{6}{5}.$$

$\therefore$ The intercept on the X-axis is $(-\frac{6}{5})$.

We see from the above examples that every equation of the first degree in two variables of the form $Ax + By + C = 0$, ($B \neq 0$), represents a line and hence it is sometimes called the **linear equation**. It can be transformed into an equivalent equation in the slope-intercept form $y = mx + h$ ($B \neq 0$). We may put

$$m = \frac{-A}{B} \text{ and } h = \frac{-C}{B} \text{ and get the required form. For instance, in}$$

Ex. 1 above, $A = 2$, $B = 3$, $C = 8$.

$$\therefore m = \frac{-A}{B} = -\frac{2}{3} \text{ and } h = \frac{-C}{B} = -\frac{8}{3}$$

$$\therefore \text{ the equation becomes } y = -\frac{2}{3}x - \frac{8}{3}.$$

EXERCISE 3.4

Write the following equations in the slope-intercept form. State the slope of the graphs in each case and also state the intercepts made by the graphs on the axes.

1. $y + x + 1 = 0$ 2. $x + y = 7$ 3. $2x - y + 8 = 0$
4. $x + 2y = 12$ 5. $3x + 5y = 9$ 6. $4x - 7y + 9 = 0$.

Complete the following statements.

7. The slope of the line $2x + 3y + 1 = 0$ is _____.
8. The intercept made by the line $\frac{x}{2} + \frac{y}{3} = 1$ on the Y-axis is _____ and that on the X-axis is _____.
9. The line $2x + 3y = 1$ cuts the X-axis in $(-, -)$ and the Y-axis in $(-, -)$.
10. If the slope of the line $5y - Ax + 7 = 0$ be $\frac{2}{5}$, then $A =$ _____.
11. The intercept made by the line $3x = 20 - 4y$ on the Y-axis is _____ and on the X-axis is _____.
12. $(3, -)$ lies on the line $2x + y = 4$.
13. The point $(4, 5)$ lies on the line $Ax = y + 3$. $\therefore A =$ _____.
14. The point $(-, -3)$ lies on the line $4y - 3x = 6$.
15. The line $y = mx + h$ makes an intercept 3 on the Y-axis and its slope is $-\frac{3}{2}$. $\therefore$ Its equation in the form $Ax + By + C = 0$ is _____.
16. The line $Ax + By + 6 = 0$ makes an intercept (-2) on the Y-axis and (-3) on the X-axis. Therefore, $A =$ _____, $B =$ _____.
17. Draw the graphs of the following equations on one and the same co-ordinate system :
(i) $2x - 3y = 5$; (ii) $3x + 2y - 5 = 0$; (iii) $2x - 5y - 7 = 0$.

§ 3.11 Graphs of Equations involving Absolute Value :

We know what is meant by the absolute value of a number. If x be a real number, then the absolute value of x is defined as follows—

When x is positive ($x > 0$) or zero, then $|x| = x$ and when x is negative ($x < 0$), then $|x| = -x$.

Therefore, the equation $y = |x|$ combines two equations

- (i) $y = x$, when $x \geq 0$ and
- (ii) $y = -x$, when $x < 0$, into one equation.

Let us prepare a table of some of the corresponding values of x and y from the equation.

$$y = |x|$$

x	0	-2	2	-4	4	6	-6	-8	8
$y = x $	0	2	2	4	4	6	6	8	8

You will notice that the values of y are all positive except in the case of $(0, 0)$. Hence, the graph must pass through the origin and lie in the first and the second quadrants only.

The graph as shown in fig. 3-12 consists of two rays whose common end-point is the origin. Draw the graph in your note-book.

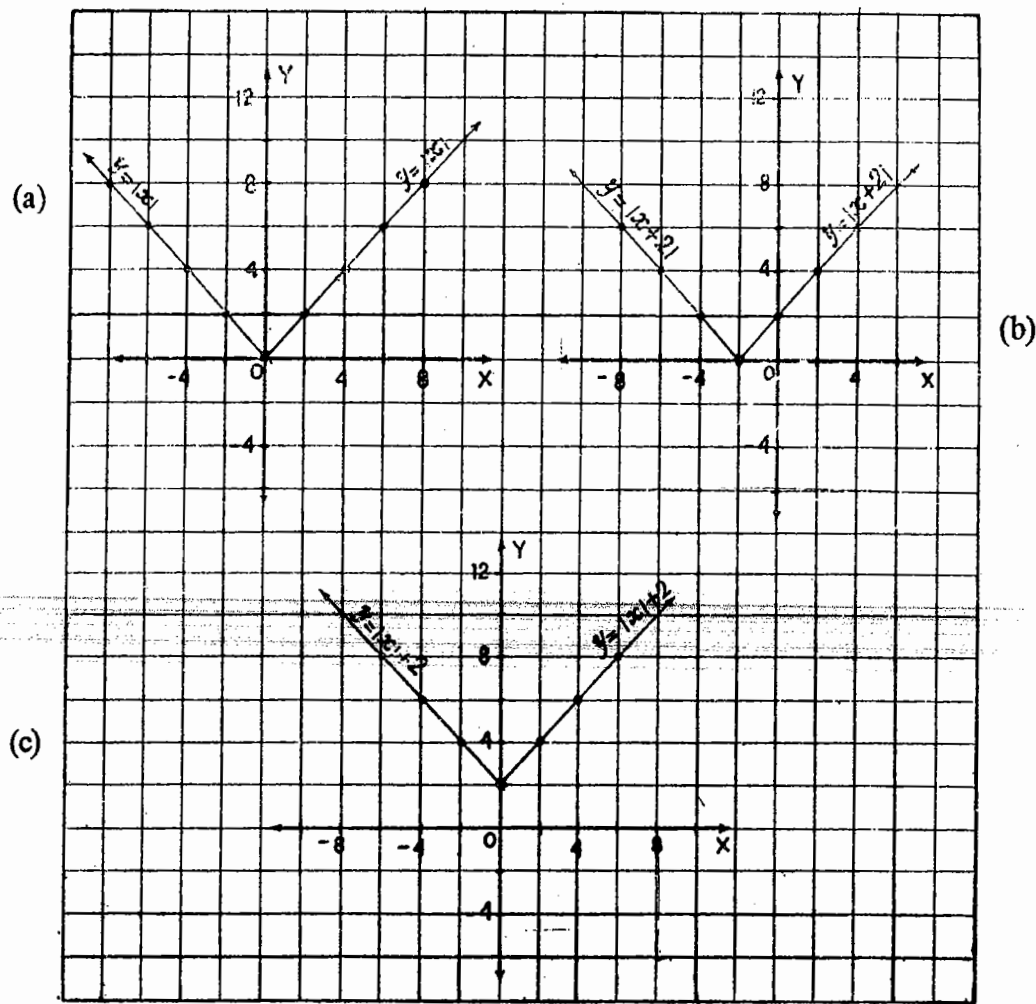


Fig. 3-12

In the fig. 3-12 above, graphs of $y = |x + 2|$ and $y = |x| + 2$ are also shown. Notice how the term '2' affects the graph of $y = |x|$ when (i) it is included in the absolute value and (ii) when it is outside the absolute value.

In $y = |x + 2|$ the effect of the term '2' in the expression of the absolute value is to shift the point $(0, 0)$ of $y = |x|$ to $(-2, 0)$, while the effect of the term '2' in (ii) is like the term ' h ' in the equation $y = mx + h$, i.e. the effect is in shifting the point $(0, 0)$, two units upwards along the Y-axis. [see Fig. 3-12 (b), (c).]

Prepare and complete the tables for the above two graphs as shown on the next page and draw the graphs in your note-book.

$$y = |x + 2|$$

x	0	-2	2	-4	4	-6	-8
$y = x + 2 $	2	0					

$$y = |x| + 2$$

x	0	2	-2	-4	4	6	-6
$y = x + 2$							

EXERCISE 3.5

Draw the graphs of the following open sentences on one and the same co-ordinate system :

1. $y = -|x|$
2. $y = 2 - |x|$
3. $y = |2 - x|$
4. $y + |x - 3| = 0$
5. $y - |x| + 3 = 0$

§ 3.12 Regions on a Plane :

In fig. 3.13 below, you see the graph of $y = 10$. The plane of the graph-paper has been separated into two half planes by the said line. The co-ordinates of any point on the line are of the form $(x, 10)$ where $x \in \mathbb{R}$, (i.e. the y -co-ordinate of every point on the line is 10, whatever be its x -co-ordinate.)

Let us name the two half planes H_1 and H_2 .

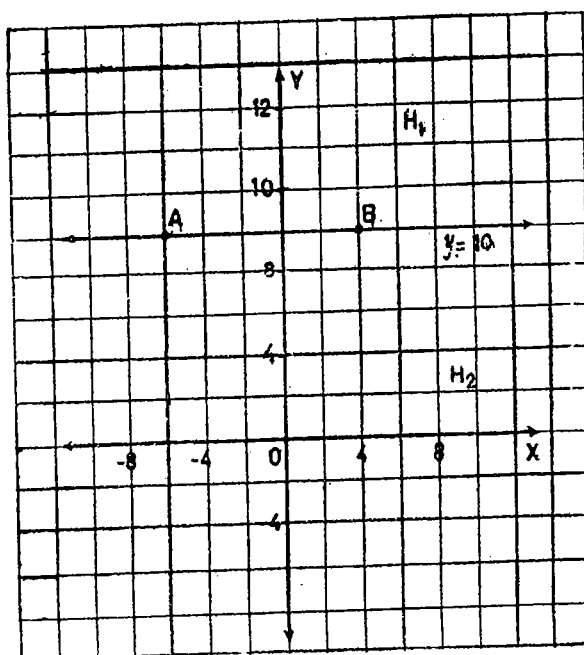


Fig. 3.13

What should be the values of the y -co-ordinates of the points in the half plane H_1 ? What should be the values of the y -co-ordinates in the half plane H_2 ?

The three disjoint sets of points in the figure have the following distinguishing features —

- (1) All the points on the line $y = 10$, have their y -co-ordinates equal to 10;
- (2) All the points in the half plane H_1 , have their y -co-ordinates greater than 10; and
- (3) All the points in the half plane H_2 have their y -co-ordinates less than 10.

The above distinguishing features can be used to describe the sets of points as follows —

- (1) $y = 10$ is the equation representing the set of all the points on the line AB ;
- (2) $y > 10$ is the open sentence representing the set of all the points that lie in the half plane H_1 ; and
- (3) $y < 10$ is the open sentence representing the set of all the points in the half plane H_2 .

We can describe the above three sets of points as sets of ordered pairs as follows —

- (1) $\{(x, y) | y = 10\}$; (2) $\{(x, y) | y > 10\}$; and (3) $\{(x, y) | y < 10\}$.

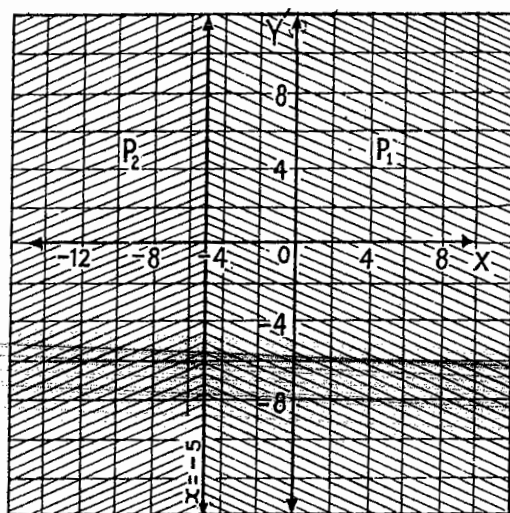


Fig. 3.14

In the figure 3.14 along side, the three sets of points into which the co-ordinate-plane is separated by the line $x = -5$ are shown —

- (1) The set of all points on the line $x = -5$;
- (2) The set of all points in the half plane P_1 is shaded as $\equiv$. The x -coordinate of all these points is greater than -5 ;
- (3) The set of all points in the half plane P_2 is shaded as $\equiv$. The x -coordinate of all these points is less than -5 .

Now, let us consider the separation of the co-ordinate—plane by the line $2x - 3y - 6 = 0$. The graph of $2x - 3y - 6 = 0$ is the line AB in the adjoining figure and the two half planes are marked R_1 and R_2 .

The co-ordinates of any point on the line AB must make the expression $2x - 3y - 6$ equal to zero. For instance, the point $(-3, -4)$ lies on AB. Substituting $x = -3$, $y = -4$ in $2x - 3y - 6$, we get, $2(-3) - 3(-4) - 6 = 0$. Similarly, the co-ordinates of the point $(6, 2)$ which also lies on the line AB, makes $2x - 3y - 6$ equal to zero.

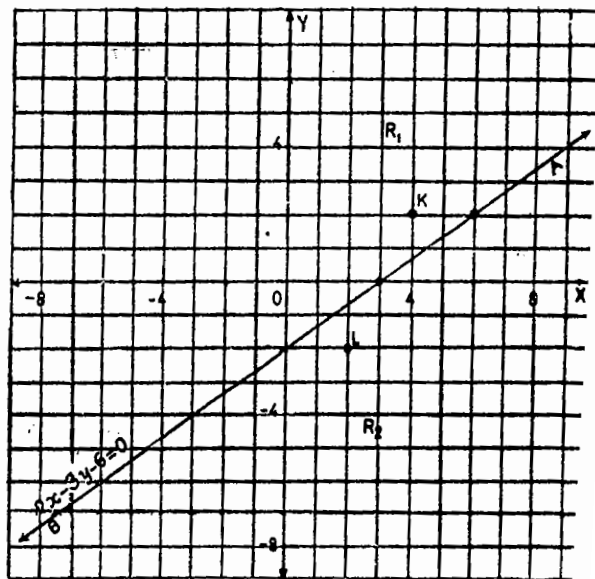


Fig. 3.15

Now, let us consider any point, say $(2, 3)$ in the half plane R_1 . If we substitute $x = 2, y = 3$ in $2x - 3y - 6$, it becomes $2(2) - 3(3) - 6 = -11$.

That is, when $x = 2, y = 3$, then $2x - 3y - 6 < 0$.

Similarly, for the point $(-3, 1)$ which also lies in R_1 , we have $2x - 3y - 6 = 2(-3) - 3(1) - 6 = -15$ which is also less than zero.

Hence, the set of all points in the half plane R_1 can be described as $\{(x, y) \mid 2x - 3y - 6 < 0\}$.

Now, consider the points $(-2, -6)$ and $(5, -5)$, both of which lie in the half plane R_2 .

For $(-2, -6)$, we have, $2x - 3y - 6 = 2(-2) - 3(-6) - 6 = 8$

That is, $2x - 3y - 6$ is positive.

For $(5, -5)$, we have, $2x - 3y - 6 = 2(5) - 3(-5) - 6 = 19$

That is, $2x - 3y - 6 > 0$.

Therefore, we may conclude that for any point in the half plane R_2 , the expression $2x - 3y - 6$ is positive. Hence, the set of points in that half plane can be described by the set of ordered pairs $\{(x, y) \mid 2x - 3y - 6 > 0\}$.

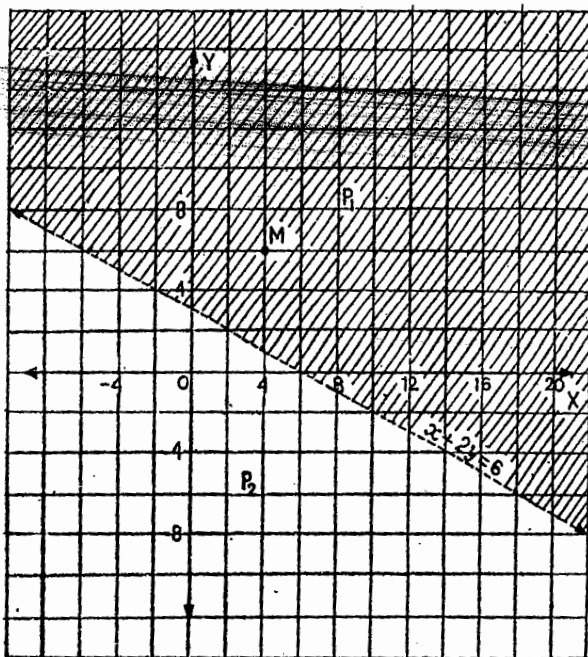


Fig. 3-16

Let us see, whether the points satisfying $x + 2y > 6$ do lie in the half plane P_1 .

Let us choose the point $(2, 4)$. Substituting $x = 2, y = 4$ in $x + 2y$, we get $2 + 2(4) = 10$ which is greater than 6.

Therefore, the set of points for which $x + 2y > 6$ is the half plane P_1 . It is shown shaded in the fig. 3-16 above.

Example 4 : Shade the region of the set of points given by $\{(x, y) \mid 2x - y \geq 5\}$.

Here, we are concerned with points whose co-ordinates satisfy (1) $2x - y = 5$ or (2) $2x - y > 5$.

Example 3 : Shade the graph of $\{(x, y) \mid x + 2y > 6\}$.

In figure 3-16 along side the line $x + 2y = 6$ is shown by the dotted line. We want the set of points for which $x + 2y > 6$. So, let us see as to which side or in which half plane such points lie. For that purpose, let us consider a point, say $(4, -3)$, which lies in the half plane marked P_2 . If we substitute $x = 4, y = -3$, in the expression $x + 2y$, we get $4 + 2(-3) = -2$ which is less than 6. Hence, the points we wish to consider do not lie in the half plane P_2 .

So, we draw the line $2x - y = 5$, as in fig. 3-17 below—

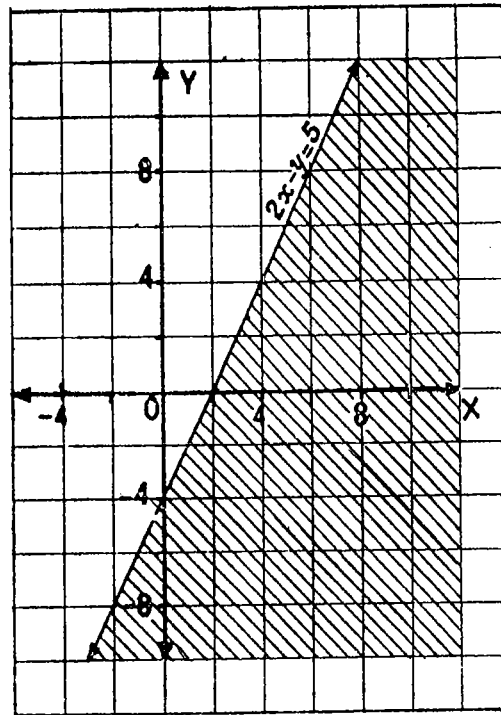


Fig. 3-17

Then we consider a point, say $(4, 1)$ in the half plane which is shaded. Substituting $x = 4$ and $y = 1$, in $2x - y$, it becomes $2(4) - 1 = 7$ which is greater than 5.

Hence, the points for which $2x - y > 5$ lie in the shaded half plane. So the graph is the set of points on the line as well as in the shaded half plane. The line is thickly drawn (not dotted as in the last example). This is to emphasise that the points which satisfy $2x - y = 5$ are included in the graph. This is also indicated by letting the shaded portion of the half plane touch this line. These two things mean that points on the line as well as in the shaded half plane are all included in the graph.

EXERCISE 3-6

- Complete the following table with reference to the fig. 3-18.

Point	Co-ordinates	Value of $2x + y$	Relation with 2	Where does the point lie
A	$(-2, 6)$	2	$= 2$	on the line
C				
D	$(-2, 3)$	-1	< 2	in H_1
E				
G				
H				
K				
L				

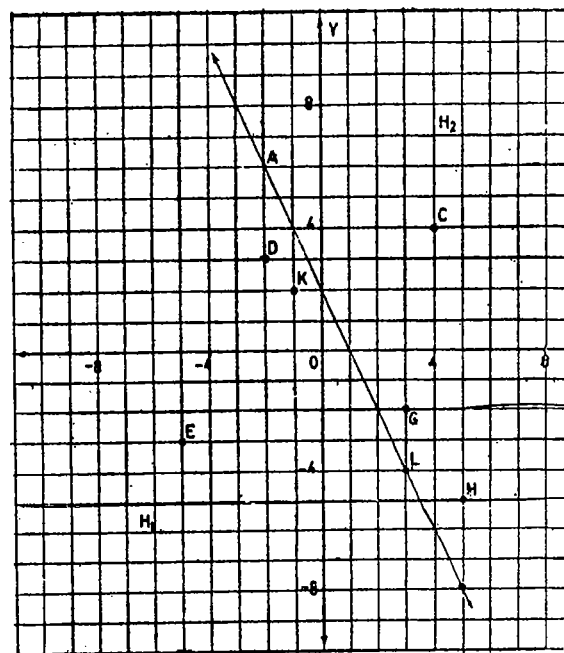


Fig. 3-18

2. If $P = \{(x, y) \mid x + 2y \leq 6\}$, shade the graph of P .

[Proceed as follows : (a) Draw the graph of $x + 2y = 6$; (b) Determine the half plane in which the points satisfy $x + 2y < 6$]

3. Each of the two points given below belong to one of the sets.

$$L = \{(x, y) \mid x < y\}$$

$$M = \{(x, y) \mid y < -x\}$$

$$N = \{(x, y) \mid y = x\}$$

Make a table as given below and complete it.

Point	Set to which it belongs	Point	Set to which it belongs
(a) (10, -15)	_____	(d) (0, 4)	_____
(b) (8, 8)	_____	(e) (0, 0)	_____
(c) (-5, -10)	_____	(f) (-15, 20)	_____

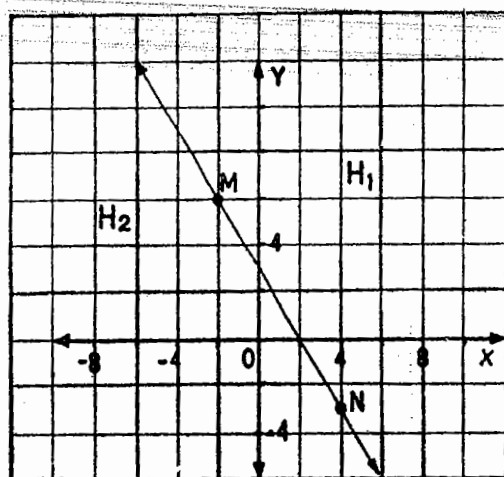


Fig. 3.19

4. In the adjoining figure 3.19, the line MN represents the graph of $3x + 2y = 6$. Which of the following sets describes the half plane H_1 ?

- (i) $\{(x, y) \mid 3x + 2y = 6\}$
- (ii) $\{(x, y) \mid 3x + 2y < 6\}$
- (iii) $\{(x, y) \mid 3x + 2y \leq 6\}$
- (iv) $\{(x, y) \mid 3x + 2y > 6\}$

5. On the same co-ordinate system shade the regions differently, represented by—

- (i) $\{(x, y) \mid x + 2y \leq 4\}$,
- (ii) $\{(x, y) \mid y - 2x \leq 4\}$.

Which set does the doubly shaded region represent?

Chapter 4 : SIMULTANEOUS EQUATIONS

§ 4.1

You know how to solve an equation'. (An equation, in fact, is an open sentence.) Broadly speaking to solve an equation is to find numbers that will make the open sentence a true statement.

To solve $x + 4 = 7$ is to find such values of x as will make it a true statement. These values of x are called the solutions of the given equation, and the set of 'all' solutions is called 'the solution set' of the equation or 'the truth set' of the open sentence. In solving an equation, we employ certain operations. We can use only those operations that will not change the truth set. With the help of such operations we reduce a given complicated equation to a simple one. To solve such an equation, then becomes a trivial thing. This is how we solved the equations, in one variable.

§ 4.2 Equation in Two Variables:

So far you were concerned with equations containing a single variable. Now, we will discuss equations containing two variables. You have considered such equations in the chapter on graphs.

How to find the solution set of $2x + y = 7$

In other words, how to obtain the truth set of this open sentence? As the equation contains two variables, x and y , we have to be careful while stating the truth set. Before we arrive at the answers to the above problems, let us try some values of x and y .

$x = 1$ and $y = 2$ give $2(1) + 2 = 7$ or $4 = 7$. but this is a false statement.

$x = 2, y = 2$ give $6 = 7$. This statement is also false.

$x = 2, y = 3$ give $7 = 7$, which is a true statement.

$x = 3, y = 1$ give $7 = 7$. This again is found to be a true statement.

However, $x = 1$ gives $2 + y = 7$, and $y = 3$ gives $2x + 3 = 7$. But these again are just open sentences. Thus, it is not enough to substitute only for one variable. For, we then obtain open sentences instead of true or false statements. Hence to obtain a statement from a given open sentence we have to substitute values for each of the variables that appear in it.

From the foregoing discussion we conclude —

(i) In order to obtain the truth set of an open sentence in two variables, it is not sufficient to substitute values for only one variable. We have to do so for both the variables.

(ii) There can be more than one pair of values of the variables that will make the given open sentence a true statement.

You have seen how to obtain, by trial and error, the values of x and y that will make the given equation or the open sentence true.

After giving some value to one of the variables, we are left with an equation in only one variable, and we know how to solve such an equation.

We have seen in the study of graphs, that we can obtain any number of pairs of values of x and y , that make the given equation or the open sentence in two variables, true. Hence, it is impossible to enlist all the elements of a solution set or the truth set of a given equation in two variables.

Thus, according to the meaning of the word 'solution' that appeals to the common sense, it is not possible 'to solve' only one equation in two variables. What is the common sense meaning of the phrase 'to solve an equation'? Normally we expect that there should be one or, at the most, only a finite number of pairs of numbers satisfying the given equation. This is not the case, as we have seen, with one equation in two variables. Therefore, for the time being, we shall keep using the phrase 'truth set' instead of 'solution set', in connection with such equations.

Let us now see how to solve two equations in two variables. We divide this problem into two parts. In the first, we shall try to understand the meaning of the phrase 'solution of the equations' and in the second we shall see 'the methods of solving them'.

Note—We will confine ourselves to equations of first degree only.

§ 4.3 Simultaneous Equations :

Observe the equations,

$$2x + y = 7 \quad \dots \quad (1) \quad 4x - y = 5 \quad \dots \quad (2)$$

We have to consider these equations simultaneously. That is, we have to find such values of x and y , as will make both the equations or the open sentences, true. When such values are found they are said to satisfy the equations simultaneously. The word 'simultaneously' is important. That is why, instead of just saying 'two equations', we say 'two simultaneous equations'.

To solve two simultaneous equations, we first find the truth sets of the two equations separately. It is clear, that the elements that are common to these sets shall make both the open sentences true. Alternatively, 'their intersection set is a required truth set'. We will call this intersection set the **solution set** of the given simultaneous equations. Every element common to the truth sets of the given simultaneous equations in two variables is a solution of the equations.

We have explained the meaning of the phrase 'two simultaneous equations'. However, in this chapter we may drop the word 'simultaneous', it being understood that the phrase 'two equations' will always mean 'two simultaneous equations'.

§ 4.4 Solution set of Simultaneous Equations :

Let us now see how to obtain the solution set of the given equations.

Example 1 : $2x + y = 7$ (1) $4x - y = 5$ (2)

$(0, 7), (1, 5), (2, 3), (-1, 9), (3, 1)$, are some elements from the truth set of $2x + y = 7$.

Again, $(0, -5), (1, -1), (2, 3), (3, 7), (-1, -9)$ are some elements from the truth set of $4x - y = 5$.

$(2, 3)$ is seen to be a common element to these two sets. Hence, $(2, 3)$ is a solution of the given equations.

You know the convention, that the first number 2 in $(2, 3)$ is the value of x , while the second number 3 is the value of y . Hence, $x = 2$ and $y = 3$ is a solution of the given equations. It is an easy matter to verify this fact by actual substitution. You can try.

Example 2 : $2x - 3y = 5$ (1) $5x + 2y = 3$ (2)

$(-1, -\frac{7}{3}), (0, -\frac{5}{3}), (1, -1), (2, -\frac{1}{3}), (\frac{5}{2}, -\frac{4}{3})$ are some elements from the truth set of the first equation and $(-1, 4), (0, \frac{3}{2}), (1, -1), (2, -\frac{7}{2}), (3, -\frac{13}{2})$ are some elements from the truth set of the second.

$(1, -1)$ is seen to be a common element. Hence $x = 1, y = -1$ is a solution of these equations.

Example 3 : $2x - 7y = -20$ (1) $6x + y = 6$ (2)

$(0, \frac{20}{7}), (-1, \frac{18}{7}), (1, \frac{22}{7}), (2, \frac{24}{7}), (-10, 0), (-\frac{13}{2}, 1), (-3, 2)$ are some elements from the truth set of the first equation, and $(0, 6), (-1, 12), (1, 0), (2, -6), (\frac{5}{3}, 1)$ are some elements from the truth set of the second.

Do you see any common element? There is none. None, at least, in those that have been chosen above. Does this, therefore, mean that there cannot be any common element in the truth sets of the equations? We know that each of these truth sets contains 'infinitely many' elements. So, it is impossible to write down all the elements. If that is the case, to find the solution by writing down the elements and choosing the common ones, may not be always possible. It was just a chance that we obtained common elements in the first two examples. On the other hand, although the equations in Example (3) were simple, we could still not strike upon any common element. In practice, we may come across much more complicated equations. This method may not always work.

There is one more drawback of this method. Suppose, after putting in a lot of efforts, we succeed in obtaining a common element, some questions still remain to be considered. Can there be any more common elements? If yes, how to obtain them? Can we claim to have solved the equations 'completely'? If there is no other solution, how do we justify that there is no other solution than the one which is obtained?

Thus, again, the method described above is seen to be impracticable.

§ 4.5 Solving the Equations :

We find that the definition, 'a common element from the truth sets of two given simultaneous equations in two variables, is a solution of the equations' may be sound theoretically; but it does not work in practice. Hence we have to search for a method that will be convenient in practice and will yield all the solutions of the given equations.

Recall that the following operations do not alter the truth set of an equation—

- (1) Adding the same number to both sides of the equation.
- (2) Subtracting the same number from both sides.
- (3) Multiplying both sides by a non-zero number.
- (4) Dividing both sides by a non-zero number.

Let us turn to the example we first considered.

$$2x + y = 7 \dots (1) \quad 4x - y = 5 \dots (2)$$

Using the above operations we shall change the equations wherein x is expressed in terms of y .

$$2x + y = 7 \quad \therefore 2x = 7 - y$$

$$\therefore x = \frac{7 - y}{2} \quad \dots (3)$$

Similarly, from equation (2), we get

$$4x - y = 5 \quad \therefore 4x = y + 5$$

$$\therefore x = \frac{y + 5}{4} \quad \dots (4)$$

It is clear that the truth sets of (1) and (3) are identical and that the truth sets of (2) and (4) are identical. For, we have performed only such operations that would not change the truth sets. Then, to obtain the common elements from the truth sets of (1) and (2) is to obtain the common elements of the truth sets of (3) and (4). In other words, a solution of the equations (1) and (2) is also a solution of the equations (3) and (4); and vice versa. Hence, we may solve the equations (3) and (4) instead of (1) and (2).

Let us now consider the equations (3) and (4) further. In order to make these equations or open sentences simultaneously true, the same value of x and the same value of y must make both the equations true. This will happen if the values of x obtained in terms of y are equal. Let us equate them and write.

$$\frac{7 - y}{2} = \frac{y + 5}{4} \quad \dots (5)$$

This is an equation in one variable, viz., y . You know how to solve it.

$$2(7 - y) = y + 5 \quad (\text{multiplying both sides by } 4)$$

$$\therefore 14 - 2y = y + 5$$

$$\therefore -3y = -9$$

$$\therefore y = 3$$

Using this in (3) or (4), we can obtain the value of x .

$$x = \frac{7-y}{2} = \frac{7-3}{2} = 2.$$

Thus $x = 2$ and $y = 3$ is a solution of the equations (3) and (4) and hence it is a solution of (1) and (2) also.

Let us now turn to the other point. While obtaining the equation (5) from (3) and (4), we presumed no special value for x , (or y). We just wanted that they should be, the same on both sides. We did not even think of the number of such values. Naturally, this method yields all the possible values. Of course, here we got only one value. Does this mean that there is no other solution ?

You know that the graph of a linear equation in two variables is a straight line. Two simultaneous equations give us two lines. These lines can intersect at the most in one point. The co-ordinates of this point constitute the solution of the given simultaneous equations. We therefore, conclude that the solution set of two simultaneous equations contains at the most one pair.

By deriving the equations (3) and (4) from (1) and (2) we expressed x in terms of y . We can as well express y in terms of x . (Perhaps this might be simpler.)

$$2x + y = 7 \quad \dots (1)$$

$$\therefore y = 7 - 2x \quad \dots (6)$$

$$\text{and } 4x - y = 5 \quad \dots (2)$$

$$\therefore -y = 5 - 4x \text{ or } y = 4x - 5 \quad \dots (7)$$

Equating the two expressions for y in (6) and (7)

$$\text{we get } 7 - 2x = 4x - 5 \quad \dots (8)$$

This gives $x = 2$. Using this value in (6) or (7), we get $y = 3$.

EXERCISE 4.1

Solve the following simultaneous equations by expressing x in terms of y or y in terms of x .

$$1. \quad 2x - y = -3; \quad 2y - x = 3 \quad 2. \quad 2x - 3y = 5; \quad 2x - y = 4$$

$$3. \quad x + y = 5; \quad 2x - y = 4 \quad 4. \quad x + y = 0; \quad 3x - 2y = 10$$

$$5. \quad x + 2y = 1; \quad 2x + 5y = 3$$

§ 4.6 To Solve Equations by Elimination :

It is not obligatory on us to solve the given simultaneous equations by expressing one of the variables in terms of the other. We can simplify our work.

We can perform certain operations without altering the truth sets of an equation. Already we have seen some such operations. We state below some more. These will be particularly useful to us in solving the simultaneous equations.

(5) Adding the two equations, (That is, adding their respective sides.)

(6) Subtracting one equation from the other. (That is, subtracting their respective sides.)

(7) Multiplying the equations, (that is, both sides) by any numbers and adding them (both sides).

Note : The operation (7) is not substantially different from the others. It can be obtained from the operations (3) and (5). We can say the same about (6) also. However, it is a matter of convenience that we have stated them separately.

Now we will carry out the operations one after another so that one of the variables disappears. This process is called **elimination**.

Let us apply these, again to the first example.

$$2x + y = 7 \quad \dots (1) \qquad 4x - y = 5 \quad \dots (2)$$

Let us eliminate x . Multiply the first equation by 2 and obtain.

$$4x + 2y = 14 \quad \dots (9)$$

Now subtract the second equation from (9).

$$\begin{array}{r} 4x + 2y = 14 \\ 4x - y = 5 \\ \hline - \quad + \quad - \\ \hline \therefore 3y = 9 \quad \dots (10) \\ \therefore y = 3 \end{array}$$

Using this in (1) or (2) as before we get $x = 2$.

We can as well eliminate y . (In some cases this may prove to be easier.)

Let us eliminate y from the same equation. We can do so by adding (1) and (2).

$$\begin{array}{r} 2x + y = 7 \\ 4x - y = 5 \\ \hline 6x = 12 \quad \dots (11) \\ \therefore x = 2 \end{array}$$

From (1) or (2) we get $y = 3$.

The method of elimination, broadly speaking, is as below —

If we want to eliminate x , then

- (1) multiply the second equation by the coefficient of x in the first;
- (2) multiply the first equation by the coefficient of x in the second;
- (3) subtract one of these new equations from the other.

However, in practice, the actual values of the coefficients may suggest some short cuts. Experience and practice will teach you how to find such short cuts.

Example 1 : Solve. $4x + 7y = 2 \quad \dots (1)$ and $3x + 5y = 1 \quad \dots (2)$

Part 1 : Let us eliminate x .

The coefficient of x in (2) is 3, and that in (1) is 4. So multiply equation (1) by 3 and (2) by 4.

We then have

$$12x + 21y = 6 \quad \dots (3)$$

$$* 12x + 20y = 4 \quad \dots (4)$$

On subtracting (4) from (3) we get,

$$12x + 21y = 6$$

$$12x + 20y = 4$$

$$\begin{array}{r} - \\ - \\ - \\ \hline y = 2 \end{array}$$

Using this value of y in (1) we get $x = -3$.

Part 2 : Putting $x = -3$ and $y = 2$ in (1) and (2),

$$\begin{array}{r|l} 4(-3) + 7(2) \stackrel{?}{=} 2 & 3(-3) + 5(2) \stackrel{?}{=} 1 \\ -12 + 14 \stackrel{?}{=} 2 & -9 + 10 \stackrel{?}{=} 1 \\ 2 \checkmark 2 & 1 \checkmark 1 \end{array}$$

Thus the solution set is $\{(-3, 2)\}$.

Example 2 : Solve. $3x - 4y = 24$ and $5y + 2x + 7 = 0$.

Part 1 : Let us express the equations as below:

$$3x - 4y = 24 \quad \dots (1) \quad 2x + 5y = -7 \quad \dots (2)$$

To eliminate y we multiply (1) by 5, (2) by 4 and add.

$$15x - 20y = 120 \quad \dots (3)$$

$$8x + 20y = -28 \quad \dots (4)$$

$$23x = 92$$

$\therefore x = 4$. Putting this in (2), we get $y = -3$.

Part 2 : Putting $x = 4$ and $y = -3$ in (1) and (2),

$$\begin{array}{r|l} 3(4) - 4(-3) \stackrel{?}{=} 24 & 2(4) + 5(-3) \stackrel{?}{=} -7 \\ 24 \checkmark 24 & -7 \checkmark -7 \end{array}$$

Thus the truth set is $\{(4, -3)\}$.

Example 3 : Solve. $\frac{1}{2}y - \frac{1}{3}x + 1 = 0$; $\frac{1}{4}y + \frac{1}{5}x$

Part 1 : Let us first remove the fractions by multiplying the first equation by 6, and second by 20 and get

$$3y - 2x = -6 \quad \dots (3)$$

$$5y + 4x = 100 \quad \dots (4)$$

It is easier to eliminate x . Multiply (3) by 2 and add it to (4).

$$6y - 4x = -12$$

$$5y + 4x = 100$$

$$11y = 88$$

$$\therefore y = 8$$

Putting this value in (4), we get $x = 15$.

Part 2 : Putting $x = 15$ and $y = 8$ in the given equations, we get

$$\begin{array}{r|l} \frac{1}{2}(8) - \frac{1}{3}(15) + 1 \stackrel{?}{=} 0 & \frac{1}{4}(8) + \frac{1}{5}(15) \stackrel{?}{=} 5 \\ 0 \checkmark 0 & 5 \checkmark 5 \end{array}$$

$\therefore$ The solution is $x = 15$ and $y = 8$.

EXERCISE 4.2

Solve the following simultaneous equations by eliminating either x or y .

1. $3x = 4y - 6$; $5x - y = 7$ 2. $2x + y + 1 = 0$; $x - 4y = 13$

3. $3x - y = 19$; $3x + y = 29$ 4. $x + 2y = 6$; $4x + 3y = 4$

5. $x + y = 7$; $2x + 3y = 18$ 6. $3x + 7y = 35$; $7x - 7y = 35$

§ 4.7 Illustrated Examples:

Example 1: Solve. $7x + 13y = 47 \dots (1)$ and $13x + 7y = 53 \dots (2)$

The coefficients of x and y appear to have been interchanged. Such equations can be reduced to simpler ones, namely, $x + y = \dots$ and $x - y = \dots$. The elimination of a variable then becomes too easy.

Part 1: Adding the equations we get,

$$7x + 13y = 47$$

$$13x + 7y = 53$$

$$20x + 20y = 100$$

$$\therefore x + y = 5 \dots (3)$$

Subtracting (2) from (1),

$$7x + 13y = 47$$

$$13x + 7y = 53$$

$$-6x + 6y = -6$$

$$\therefore -x + y = -1 \dots (4)$$

Adding (3) and (4),

$$x + y = 5$$

$$-x + y = -1$$

$$2y = 4$$

$$\therefore y = 2$$

Putting this value in (3) we get, $x = 3$.

Part 2: Putting $x = 3$ and $y = 2$ in the given equations,

$7(3) + 13(2) \stackrel{?}{=} 47$ $47 \checkmark = 47$	$13(3) + 7(2) \stackrel{?}{=} 53$ $53 \checkmark = 53$
---	---

$\therefore$ The solution = (3, 2)

Example 2 : Solve. $17x - 29y = 53$ and $29x - 17y + 7 = 0$

Part 1: $17x - 29y = 53$... (1)

$29x - 17y = -7$... (2)

Adding (1) and (2)

$17x - 29y = 53$

$29x - 17y = -7$

$46x - 46y = 46$

$\therefore x - y = 1$... (3)

Subtracting (1) from (2) we get,

$12x + 12y = -60$

$\therefore x + y = -5$... (4)

Adding (3) and (4) we get,

$2x = -4$

$\therefore x = -2$

Putting this value in (3), we get $y = -3$.

Part 2 : Putting $x = -2$ and $y = -3$ in the given equations

$17(-2) - 29(-3) \stackrel{?}{=} 53; \quad 29(-2) - 17(-3) + 7 \stackrel{?}{=} 0$

$-34 + 87 \stackrel{?}{=} 53; \quad -58 + 51 + 7 \stackrel{?}{=} 0$

$53 \checkmark 53 \quad 0 \checkmark 0$

$\therefore$ The solution set = $\{(-2, -3)\}$

EXERCISE 4.3

Solve the following equations.

1. $32x + 33y = 31$; $33x + 32y = 34$
2. $13x + 18y = 80$; $18x + 13y = 75$
3. $12x + 17y = 53$; $17x + 12y = 63$
4. $15x - 17y = 28$; $15y - 17x + 36 = 0$
5. $23x - 29y = 98$; $23y + 110 = 29x$.

§ 4.8 Equations and Graphs :

We have learnt how to draw graphs of linear equations in two variables. We find some elements in the truth sets of a given equation. These elements are the ordered pairs of real numbers. The first number in such an element is the value of x and the second number is the value of y . With these numbers, we plot the points on a graph paper. The first number in an ordered pair of real numbers is the x -co-ordinate of the point, and the second number is the y -co-ordinate. The set of 'all' points that can be obtained from 'all' the pairs in the truth set of a given equation is the graph of the equation (of course, it is impossible to plot 'all' such points). You have seen that the graph of a first degree equation is a line. This line is the graph of the equation.

Mind well the meaning of the statement that a line is the graph of a given equation.

- (1) Every point obtained from the elements (the ordered pairs) lies on the line.

- (2) The ordered pair of the co-ordinates of every point on the line satisfies the equation, or belongs to its truth set.

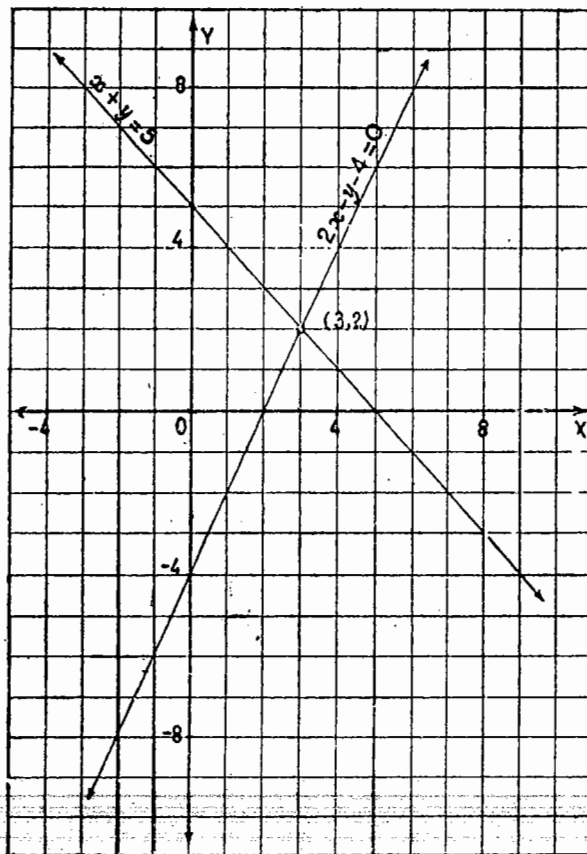


Fig. 4.1

Hence, there cannot be any other ordered pair in the solution set. Thus the solution set of the given equation is $(3, 2)$.

Example 2 : Solve graphically
 $x - y = 5$; $2x - 3y = 12$.

$x - y = 5$ and $2x - 3y = 12$ are lines and they intersect in the point $(3, -2)$ as can be seen from Fig. 4.2. The ordered pair $(3, -2)$ can be seen to make both equations true. No point other than $(3, -2)$ lies on the graphs of both the equations. Hence there cannot be any other ordered pair in the solution set. Thus the solution of the given equations is $(3, -2)$.

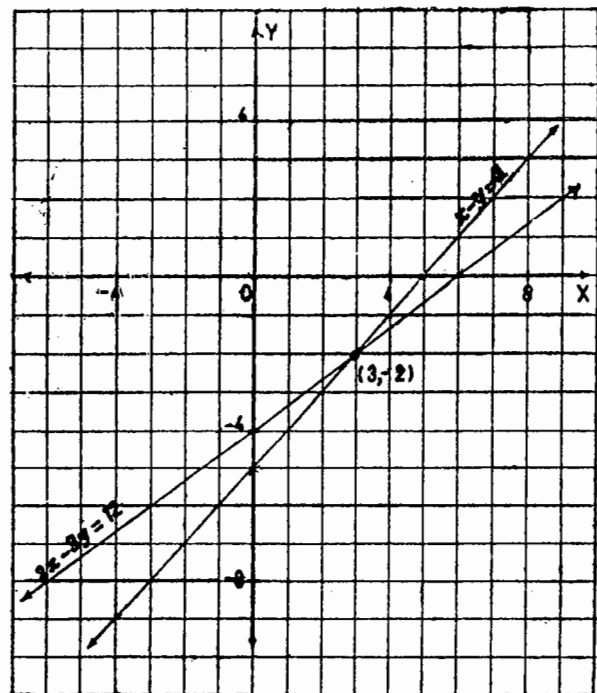


Fig. 4.2

Once these two points are well understood, it is a simple matter to solve the equation by graphs.

We know that the solution of two simultaneous equations is the common element in the truth sets of both. As far as graphs are concerned this solution means the co-ordinates of the point of intersection of the graphs of the equation.

Example 1: Solve $x + y = 5$ and $2x - y - 4 = 0$ by means of graph.

The graphs $x + y = 5$ and $2x - y - 4 = 0$ are straight lines and they intersect in the point $(3, 2)$ as can be seen from Fig. 4.1. The ordered pair $(3, 2)$ can be seen to make both the equations true. No point other than $(3, 2)$ lies on the graphs of both the equations.

EXERCISE 4.4

Solve the following equations by means of graph.

1. $x + y = 7$; $4x + 3y = 25$.
2. $4x + 3y = 12$; $5x + 8y = 15$.
3. $x - y = 5$; $2x - 3y - 12 = 0$.
4. $3y - x - 9 = 0$; $5y + 3x - 1 = 0$.
5. $x = 1 + y$; $2x + 4y = 17$.
6. $3x + 2y = 6$; $y = 3x + 3$.

§ 4.9 Consistent and Inconsistent Equations :

We have said that sometimes two equations in two variables 'cannot be solved'. Let us examine this statement with the help of graphs.

While solving two equations we draw their graphs. As these graphs are lines, the co-ordinates of points of intersection of these lines is the solution of the equations.

As regards two (distinct) lines in a plane, there can be two alternatives.

- (1) Either they intersect each other, (2) or they do not intersect. (i.e., they are parallel.)

We can then conclude that the equations can have a solution whenever their lines intersect. On the other hand, when their lines are parallel, they do not intersect, and hence, their equations do not have any solution.

Two equations are said to be consistent when they possess a solution. And they are said to be inconsistent when they do not have any solution.

Graphs show that two equations will be inconsistent if their lines are parallel, i.e., if their slopes are equal. You know how to find the slope of a line from its equation.

Fig. 4.3 shows the graphs of the equations $y - 2x = 4$ and $2y = 4x - 7$. These lines are parallel could be seen just by the inspection of the equations. Express the equations in the Slope-Intercept Form : $y = 2x + 4$ and $y = 2x - \frac{7}{2}$

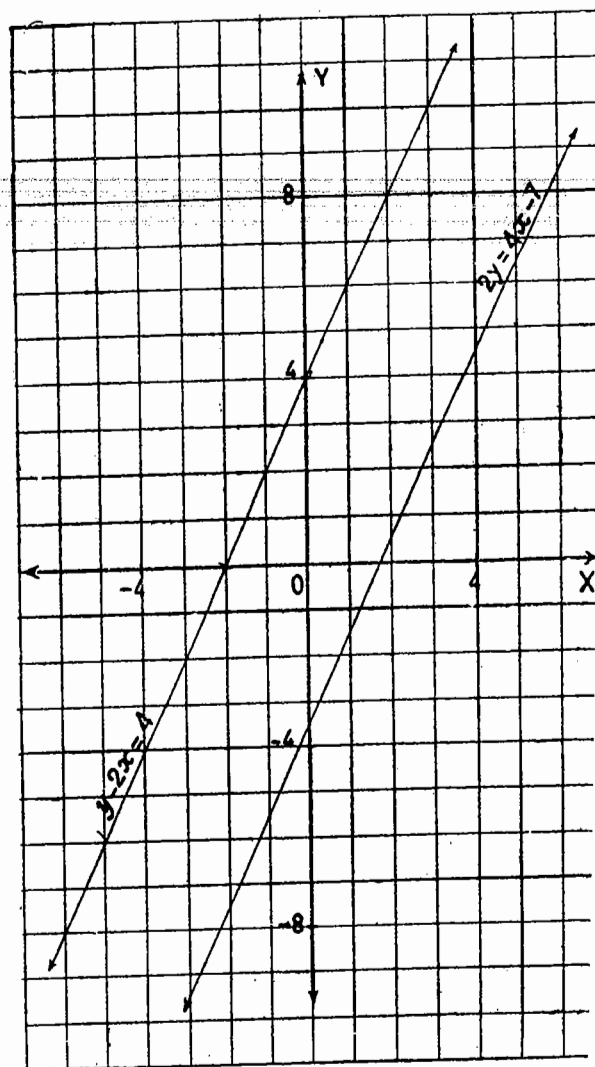


Fig. 4.3

In this form, the coefficients of x in both are equal. Hence the slopes of their graphs are equal. Of course, as the constant terms are distinct, the graphs are parallel lines. We can conclude this from the equations.

Note : In order that the lines of two equations be parallel, their slopes have to be equal. However, it can be the case that two given equations have the same lines as graph. In such a case, the equations are **not** said to be inconsistent. We will not deal with such situations in this book.

§ 4.10 Application of Simultaneous Equations :

We know how to use an equation in one variable to solve problems. To solve a problem algebraically we construct an equation on the basis of the conditions of the given problem. Sometimes we have to use two or more variables. In that case we must be able to construct as many equations as there are variables.

Let us now solve some problems with the aid of two equations in two variables.

Example 1: The perimeter of a rectangle is 40 cm and its length is longer than twice its breadth by 2 cm. Find its dimensions.

Let the length of the rectangle be x cm, and the breadth be y cm. Hence its perimeter is $2(x + y)$ cm. Its value has been given, and it gives us the first equation

$$2(x + y) = 40 \quad \dots (1)$$

The second condition in the problem gives us another equation;

$$x - 2y = 2 \quad \dots (2)$$

From (1) and (2)

$$x + y = 20 \quad \dots (3)$$

$$\text{and} \quad x - 2y = 2 \quad \dots (4)$$

Solving (3) and (4) we get $x = 14$ and $y = 6$.

Thus the length of the rectangle is 14 cm and its breadth is 6 cm.

Verification:

$$(1) \text{ The perimeter} = 2(14 + 6) = 40.$$

$$(2) \text{ Twice the breadth together with 2}$$

$$= 2 \times 6 + 2 = 14.$$

$$= \text{length}$$

Thus the conditions are fulfilled.

Ex. 2: In an election of the two candidates, the successful candidate secured 88 votes more than his rival. Had the rival candidate obtained $\frac{1}{7}$ of the total votes polled by the successful one in addition to his own votes, he would have won the election by 18 votes. Find the number of votes polled by each.

Let x be the number of votes polled by the successful candidate, and y , be the number of votes polled by the other. As the successful candidate secured 88 votes more, x is greater than y by 88.

$$\text{Therefore, } x - y = 88 \quad \dots (1)$$

Now, imagine that the votes of the successful candidate have been reduced by $\frac{1}{7}$ of his votes, i.e., by $\frac{x}{7}$, and have been given to the other one. So that they are

assumed to have polled $\left[x - \left(\frac{x}{7} \right) \right]$ and $\left[y + \left(\frac{x}{7} \right) \right]$ votes respectively. But this latter number is given to be larger than the former by 18.

$$\text{Therefore, } \left[y + \left(\frac{x}{7} \right) \right] - \left[x - \left(\frac{x}{7} \right) \right] = 18$$

$$\text{or, } 7y - 5x = 126 \quad \dots (2)$$

Solving (1) and (2), we have $x = 371$ and $y = 283$.

Thus, the successful candidate polled 371 votes and the rival candidate polled 283 votes.

Verification :

(1) Successful candidate polled $(371 - 283) = 88$ votes more than the rival. This is in agreement with the given information.

(2) $\frac{1}{7}$ of 371 is 53. The votes of the successful candidate if reduced by 53, would be $371 - 53 = 318$. While the rival candidate then gets $283 + 53 = 336$ votes. Thus he wins by $336 - 318 = 18$ votes as given in the problem.

Our solution is thus correct.

EXERCISE 4.5

1. The sum of two numbers is 55 and their difference is 15. Find the numbers.
2. The sum of the ages of Ahmad and his father is 89 years. 11 years hence, his father will be twice as old as he would be then. Find their present ages.
3. The sum of a two digit number and the number obtained by interchanging its digits is 143. If the digit in the unit's place of the given number is larger than that in the tenth's place by 3, find the number.
4. A student buys 12 notebooks of 80 pages each and 8 notebooks of 120 page: each. On paying Rs. 10 he receives 80 Paise back. If he had bought 8 note books of 80 pages each and 12 notebooks of 120 pages each, he would hav got only 20 Paise back. Find the price of each kind.
5. The perimeter of a rectangle is 46 cm. If the length were 3 cm shorter and th breadth 3 cm longer, its area would be 12 sq. cm more. Find the dimensior of the rectangle.

6. Two trains start from the same station and move in opposite directions. The speed of one of them is greater than that of the other by 15 km. per hour. If after 3 hours, they are found to be 495 km. apart, find their speeds.
7. On the first day in a X class, $\frac{2}{7}$ of the total number of students in the class were found to be boys. Next day, when 3 more girls and 5 more boys were admitted, the girls were found to be $\frac{3}{10}$ of the total number. Find the number of girls in the class on the first day.
8. A factory has 500 workers including skilled as well as unskilled labourers. A skilled worker receives Rs. 8/- per day, while an unskilled one receives Rs. 6/- per day. If the total salary of the day amounts to Rs. 3,352/- find the number of skilled workers.
9. Three apples and twelve oranges cost Rs. 4.80 while two apples and five oranges cost Rs. 2.30 only. Find the rate of apples per dozen.
10. The difference between the measures of the acute angles of a right-angled triangle is 14. Find their measures.
11. For a 'Janata Show' of a comedy 1,000 tickets were sold for a total of Rs. 1,300. If the rates of the tickets were Re. 1/- and Rs. 2/- each, how many of each kind were sold ?
12. Where should the fulcrum be placed on a 1.2 decimetres bar if it is to balance with 1.4 kg. weight on one end and a 4.2 kg. weight on the other end ?
13. A friend of yours came out of a post office having spent Rs. 4.40 for total of thirty tickets worth 10 paise and 20 paise. How many of each type did he buy ?
14. A change for Rs. 34/- was taken from the Reserve Bank in ten-paise and 25 paise coins. If the total number of coins was 220, how many coins of each type were asked for ?
15. An airplane made a 960 km. trip with the wind in 1 hr. and 20 min. Returning against the wind required an hour and a half. Find the cruising speed of the plane in still air.
16. The difference between two numbers is 80 and five times the smaller is equal to four times the larger. Find the numbers.
17. The sum of two numbers is 560. Three times one of them is four times the other. Find the numbers.
18. Seven chairs and three tables cost Rs. 610/- If five tables and six chairs are purchased the cost would increase by Rs. 180/- Find the cost of each chair.
19. It takes me 8 hours to reach a town B from town A. If I had increased my speed by 8 km. p.h. I would have completed the journey one hour earlier. What is the distance between the two towns ?
20. The difference between the measures of a pair of opposite angles of a cyclic quadrilateral is 8. Find their measures.

Chapter 5 : INDICES

So far you have learnt to deal with integral indices, i.e., numbers involving indices which are either positive integers, negative integers or zero. You are now going to study the algebra of numbers with (fractional) rational indices. Before we proceed to do this, we shall have some revision.

§ 5.1 Revision :

(1) If m is a natural number then, $a^m = a \times a \times a \times \dots m \text{ times}$. a is called the 'base' and m is called the index or exponent. a^m is called the m th power of a . a^m is read as 'a raised to the power m '.

(2) If $a \neq 0$ then $a^0 = 1$.

(3) If $a \neq 0$ then $a^{-r} = \frac{1}{a^r}$ and $a^r = \frac{1}{a^{-r}}$.

(4) The laws of indices for positive integers, negative indices and zero are as follows—

(i) $a^m \times a^n = a^{m+n}$ Product of the powers.

(ii) $a^m \div a^n = a^{m-n}$ Quotient of the powers.

(iii) $(a^m)^n = a^{mn}$ Power of the power.

(iv) $(ab)^m = a^m b^m$ Power of the product.

(v) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ Power of the quotient.

EXERCISE 5.1

Fill in the blanks in the following statements with appropriate words/numbers.

1. $x^{13} = x \times x \times x \times \dots$ times.

2. $7 \times 7 \times 7 \times \dots$ 10 times =

3. In the expression x^a , x is called and a is called.

4. $a^{-3} = \frac{1}{a^k}$. $\therefore k =$

5. $2^7 = \frac{1}{2^p}$. $\therefore p =$

6. $(xy^2)^0 \times (5^3 z^4)^0 =$

7. $x^7 \times x^{-3} \times x^0 =$

8. $a^{-13} \div a^{-4} = a^k$. $\therefore k =$

9. $\frac{2^8 \times 2^{-3}}{2^{-4}} = 2^m$. $\therefore m =$

10. $\frac{5^{-2}}{5^3 \times 5^0} = 5^p$. $\therefore p =$

11. $(2^7)^{-4} = 2^m$. $\therefore m =$

12. $(y^{-3})^{-5} = y^n$. $\therefore n =$

13. $(b^{-3})^7 = b^k$. $\therefore k =$

14. $(2 \times x)^{-3} = 2^m \times x^n$. $\therefore m =$, $n =$

15. $\left(\frac{x}{y}\right)^7 = \frac{x^m}{y^n}$. $\therefore m =$, $n =$

$$16. (x^{-2} y^3)^3 = x^m y^n. \therefore m = \text{---}, n = \text{---}$$

$$17. \left(\frac{a^2 y^{-3} b^5}{b^{-1} x^2} \right)^4 = a^m b^n x^k y^p.$$

$$\therefore m = \text{---}, n = \text{---}, k = \text{---}, p = \text{---}.$$

§ 5.2 Fractional Indices :

Study the formulae given below —

$$(i) x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \quad (ii) PV^{1.4} = K.$$

We observe that in these formulae fractional indices have been used. What meaning can be assigned to numbers with fractional indices ? We know the meaning of a^m , when m is a positive integer. Can we assign the same meaning to a^m , when m is a fraction ?

We know that, $x^7 = x \times x \times \dots$ 7 times. But can $x^{\frac{3}{2}}$ mean $x \times x \times \dots$ $\frac{3}{2}$ times ? Multiplying x with itself $\frac{3}{2}$ times is obviously meaningless. Hence it is evident that the meaning assigned to a^m , when m is a positive integer, does not apply to a^m , when m is a fraction.

While assigning meaning to a^m , when m is a fraction, it should be ensured that the laws of indices which were true for positive integral, zero and negative integral indices, continue to hold good when the indices are fractional numbers. After assigning the meaning to a^m , when m is a fraction, the following laws of indices which are true when indices are positive integers, zero and negative integers, should continue to hold in the case of the numbers having fractional indices —

$$(i) a^m \times a^n = a^{m+n} \quad (ii) a^m \div a^n = a^{m-n} \quad (iii) (a^m)^n = a^{mn}$$

$$(iv) (ab)^m = a^m b^m \quad (v) \left(\frac{a}{b} \right)^m = \frac{a^m}{b^m}$$

To achieve this objective, let us assume the first law of indices ($a^m \times a^n \times a^p \times \dots = a^{m+n+p+\dots}$) to be true even when the indices are fractional. This will help us to assign meaning to a^m , when m is a fraction. After assigning meaning to a^m in this way when m is a fraction, we shall see that the other laws of indices also hold good.

§ 5.3 Meaning of $x^{p/q}$. (p, q are integers, $q \neq 0$) :

Last year in the chapter on Radicals, you have studied that the square root, cube root, fourth root, fifth root etc. of a real number are represented by using radical sign as $\sqrt[2]{}$ (or $\sqrt{}$), $\sqrt[3]{}$, $\sqrt[4]{}$, $\sqrt[5]{}$ etc. respectively.

For example :

$$(i) \text{ Square root of 7 is represented as } \sqrt[2]{7} \text{ (or } \sqrt{7}).$$

$$(ii) \text{ Square root of } x \text{ is represented as } \sqrt[2]{x} \text{ (or } \sqrt{x}).$$

(Note: In case of square roots only, the number 2 showing the order of the radical sign can be omitted i.e. $\sqrt{a}$ means $\sqrt[2]{a}$).

(iii) Cube root of 13 is represented as $\sqrt[3]{13}$.

(iv) Fifth root of 37 is represented as $\sqrt[5]{37}$.

(v) n th root of x is represented as $\sqrt[n]{x}$.

You also know that, $\sqrt{2}$ is a number whose square is 2; $\sqrt[5]{37}$ is that number whose fifth power is 37 etc. Hence, square, cube, fourth power, fifth power, n th power etc. of the square root, cube root, fourth root, fifth root, n th root respectively of a given real number is the number itself. e.g. (i) $(\sqrt{7})^2 = 7$, (ii) $(\sqrt[3]{4})^3 = 4$,

(iii) $(\sqrt[5]{13})^5 = 13$, (iv) $(\sqrt[n]{x})^n = x$, (v) $(\sqrt[k]{17})^k = 17$, (vi) $(\sqrt[n]{x})^n = x$.

Now, study the following examples. They will assist us in assigning meaning to the numbers having fractional indices.

Assuming the first law of indices ($a^m \times a^n \times a^p \times \dots = a^{m+n+p+\dots}$) to be true for all rational values of the indices, let us find the meaning of $(5^{\frac{1}{3}})^3$

$$\begin{aligned} (5^{\frac{1}{3}})^3 &= 5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}} && \dots (\text{Definition of } a^m). \\ &= 5^{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} && \dots (a^m \times a^n = a^{m+n} \text{ for all rational values of } m \text{ and } n) \\ &= 5^1 \\ &= 5 \end{aligned}$$

$\therefore (5^{\frac{1}{3}})^3 = 5$. It means $5^{\frac{1}{3}}$ is a number whose cube is 5. But the number whose cube is 5, is represented by $\sqrt[3]{5}$. Hence, we assign the following meaning to $5^{\frac{1}{3}}$.

$$5^{\frac{1}{3}} \text{ is cube root of } 5. \therefore 5^{\frac{1}{3}} = \sqrt[3]{5}.$$

Thus $5^{\frac{1}{3}}$ (the number having fractional power) is $\sqrt[3]{5}$ i.e. cube root of 5.

Similarly, we can assign meanings to other numbers having fractional indices.

(i) Meaning of $2^{\frac{4}{5}}$.

$$\begin{aligned} (2^{\frac{4}{5}})^5 &= 2^{\frac{4}{5}} \times 2^{\frac{4}{5}} \times 2^{\frac{4}{5}} \times 2^{\frac{4}{5}} \times 2^{\frac{4}{5}} && \dots (\text{Definition of } a^m). \\ &= 2^{\frac{4}{5} + \frac{4}{5} + \frac{4}{5} + \frac{4}{5} + \frac{4}{5}} && \dots (a^m \times a^n = a^{m+n} \text{ for all rational values of } m \text{ and } n) \\ &= 2^{\frac{4}{5} \times 5} \\ &= 2^4 \end{aligned}$$

$\therefore (2^{\frac{4}{5}})^5 = 2^4$. It means that $2^{\frac{4}{5}}$ is a number whose fifth power is 2^4 . But, the number whose fifth power is 2^4 , is represented by $\sqrt[5]{2^4}$. Hence, $2^{\frac{4}{5}}$ is the same number as $\sqrt[5]{2^4}$. $\therefore 2^{\frac{4}{5}} = \sqrt[5]{2^4}$.

(ii) Meaning of $a^{-\frac{2}{3}}$.

$$\left(a^{-\frac{2}{3}}\right)^3 = a^{-\frac{2}{3}} \times a^{-\frac{2}{3}} \times a^{-\frac{2}{3}}$$

$$= a^{-\frac{2}{3} + (-\frac{2}{3}) + (-\frac{2}{3})}$$

$$= a^{-\frac{2}{3} \times 3}$$

$$= a^{-2}$$

....(Definition of a^m).

....($a^m \times a^n = a^{m+n}$ for all rational values of m and n .)

$\therefore \left(a^{-\frac{2}{3}}\right)^3 = a^{-2}$. Hence, $a^{-\frac{2}{3}}$ is that number whose cube is a^{-2} . But

the number whose cube is a^{-2} is represented by $\sqrt[3]{a^{-2}}$.

$$\therefore a^{-\frac{2}{3}} = \sqrt[3]{a^{-2}}.$$

The procedure used in assigning proper meaning to $5^{\frac{1}{3}}$, $2^{\frac{4}{5}}$ and $a^{-\frac{2}{3}}$ will enable us to give meaning to $a^{p/q}$ also, $p \in \mathbb{I}$ and $q \in \mathbb{N}$.

(iii) Meaning of $a^{p/q}$.

The fraction p/q is either positive or negative. To cover both of these cases we choose p to be an integer and q to be a natural number.

$$(a^{p/q})^q = a^{p/q} \times a^{p/q} \times a^{p/q} \times \dots q \text{ times,} \dots \text{(By definition of } a^m \text{).}$$

$$= a^{p/q + p/q + p/q + \dots q \text{ times,} \dots} \text{(} a^m \times a^n = a^{m+n} \text{ for all rational values of } m \text{ and } n \text{).}$$

$$= a^{p/q \times q}$$

$$= a^p.$$

$\therefore (a^{p/q})^q = a^p$. This means that $a^{p/q}$ is that number whose q th power is a^p . But, we know that the number whose q th power is a^p is represented by

$\sqrt[q]{a^p}$. Hence, $a^{p/q}$ is the same number as $\sqrt[q]{a^p}$. Therefore we have to assign the

meaning $\sqrt[q]{a^p}$ to $a^{p/q}$.

$$\therefore a^{p/q} = \sqrt[q]{a^p}.$$

Thus, assuming the first law of indices to be true for all rational values of indices, we have been able to assign the following meaning to the numbers having fractional indices.

$$x^{p/q} = \sqrt[q]{x^p}, \quad p \in \mathbb{I} \text{ and } q \in \mathbb{N}.$$

Now, using the meaning $x^{p/q} = \sqrt[q]{x^p}$, we can write $5^{\frac{2}{3}} = \sqrt[3]{5^2}$, $7^{-\frac{2}{5}} = \sqrt[5]{7^{-2}}$, $(-8)^{\frac{2}{3}} = \sqrt[3]{(-8)^2}$ etc. Similarly, $(-4)^{\frac{3}{2}} = \sqrt{(-4)^3} = \sqrt[2]{-64}$. But the even root of negative real number, is not a real number. Since, we are dealing with only real numbers, even root of negative number has no meaning.

Therefore $x^{p/q} = \sqrt[q]{x^p}$ if p and q are integers, but when q is an even natural number, x is not negative. To avoid such complications in case of numbers having rational indices, we shall assume that the base is positive.

Now using $x^{p/q} = \sqrt[q]{x^p}$, let us express the numbers having rational indices in the form of the numbers with radical signs.

$$(i) 7^{\frac{3}{5}} = \sqrt[5]{7^3} \quad (ii) 2^{\frac{1}{3}} = \sqrt[3]{2} \quad (iii) 5^{-\frac{2}{3}} = \sqrt[3]{5^{-2}}$$

$$(iv) 9^{-\frac{1}{2}} = \sqrt[2]{9^{-1}} \quad (v) x^{-\frac{8}{7}} = \sqrt[7]{x^{-8}} \quad (vi) a^{\frac{1}{5}} = \sqrt[5]{a}$$

Similarly let us express the numbers having radical sign in the form of the numbers with an index.

$$(i) \sqrt[4]{2^3} = 2^{\frac{3}{4}} \quad (ii) \sqrt[3]{7} = 7^{\frac{1}{3}} \quad (iii) \sqrt[5]{x^{-2}} = x^{-\frac{2}{5}}$$

$$(iv) \sqrt{a^3} = a^{\frac{3}{2}} \quad (v) \sqrt[r]{x^a} = x^{\frac{a}{r}} \quad (vi) \sqrt[4]{b} = b^{\frac{1}{4}}$$

EXERCISE 5.2

1. Express the following numbers using the radical sign.

$$(i) 12^{\frac{2}{5}} \quad (ii) (23)^{\frac{1}{3}} \quad (iii) 7^{-\frac{2}{3}} \quad (iv) 9^{\frac{1}{2}} \quad (v) 29^{\frac{2}{3}} \quad (vi) a^{\frac{2}{3}} \quad (vii) x^{-\frac{m}{3}}$$

$$(viii) m^{p/q} \quad (ix) \left(\frac{m}{n}\right)^{\frac{7}{2}} \quad (x) x^{-\frac{11}{2}} \quad (xi) \text{Fourth root of } 31 \quad (xii) \text{Seventh root of } 32 \quad (xiii) \text{Cube root of } 12.$$

2. Write the following numbers in the index (i.e. exponential) form.

$$(i) \sqrt[4]{2^3} \quad (ii) \sqrt[7]{8^3} \quad (iii) \sqrt[5]{\left(\frac{3}{7}\right)^2} \quad (iv) \sqrt{x^{-3}} \quad (v) \sqrt[3]{x^{-7}} \quad (vi) \sqrt[3]{13} \quad (vii) \sqrt{27}$$

$$(viii) \sqrt[6]{\left(\frac{4}{3}\right)^4} \quad (ix) \sqrt[q]{x^p} \quad (x) \sqrt[6]{\left(\frac{x}{y}\right)^9} \quad (xi) \text{Seventh root of } 137.$$

$$(xii) \text{Square root of } 17. \quad (xiii) \text{Cube root of } 24. \quad (xiv) \text{Fourth root of } 30.$$

Now, using $x^{p/q} = \sqrt[q]{x^p}$, let us find the values of the numbers having fractional indices.

$$(i) 8^{\frac{2}{3}} = \sqrt[3]{8^2} \quad (ii) 32^{\frac{2}{5}} = \sqrt[5]{32^2} \quad (iii) 27^{\frac{2}{3}} = \sqrt[3]{27^2}$$

$$= \sqrt[3]{64} \quad = \sqrt[5]{1024} \quad = \sqrt[3]{729}$$

$$= 4 \quad = 4 \quad = 9$$

$$(iv) \left(\frac{1}{27}\right)^{-\frac{2}{3}} = \sqrt[3]{\left(\frac{1}{27}\right)^{-2}} \quad (v) \left(\frac{9}{4}\right)^{\frac{3}{2}} = \sqrt[2]{\left(\frac{9}{4}\right)^3}$$

$$= \sqrt[3]{\frac{1^{-2}}{27^{-2}}} \quad = \sqrt[2]{\frac{729}{64}}$$

$$= \sqrt[3]{\frac{27^2}{1^2}} \quad = \frac{27}{8}$$

$$= \sqrt[3]{729} \quad (vi) 64^{\frac{1}{8}} = \sqrt[8]{64}$$

$$= 9 \quad = 2$$

$$\begin{aligned} \text{(vii)} \quad 9^{\frac{1}{2}} \times 8^{\frac{1}{3}} &= \sqrt{9} \times \sqrt[3]{8} \\ &= 3 \times 2 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \text{(viii)} \quad (8^{\frac{2}{3}})^2 \times (16^{\frac{3}{4}})^3 &= (\sqrt[3]{8^2})^2 \times (\sqrt[4]{16^3})^3 \\ &= (\sqrt[3]{64})^2 \times (\sqrt[4]{4096})^3 \\ &= 4^2 \times (64)^3 \\ &= 2^4 \times 2^{18} \\ &= 2^{22} \end{aligned}$$

$$\begin{aligned} \text{(ix)} \quad \frac{27^{\frac{2}{3}} \times 8^{\frac{2}{3}}}{9^{\frac{3}{2}}} &= \frac{\sqrt[3]{27^2} \times \sqrt[3]{8^2}}{\sqrt[2]{9^3}} = \frac{\sqrt[3]{729} \times \sqrt[3]{64}}{\sqrt[2]{729}} \\ &= \frac{9 \times 4}{27} = \frac{4}{3} \end{aligned}$$

EXERCISE 5.3

1. Write the following numbers using radical sign and find their values.

$$\text{(i)} \quad 25^{\frac{3}{2}} \quad \text{(ii)} \quad 16^{\frac{1}{4}} \quad \text{(iii)} \quad 32^{\frac{3}{5}} \quad \text{(iv)} \quad \left(\frac{49}{25}\right)^{\frac{3}{2}} \quad \text{(v)} \quad 4^{\frac{3}{2}}$$

$$\text{(vi)} \quad 9^{-\frac{3}{2}} \quad \text{(vii)} \quad 8^{-\frac{2}{3}} \quad \text{(viii)} \quad \left(\frac{4}{9}\right)^{\frac{3}{2}} \quad \text{(ix)} \quad \left(\frac{4}{9}\right)^{-\frac{3}{2}}$$

$$\text{(x)} \quad 64^{\frac{1}{6}} \quad \text{(xi)} \quad 625^{\frac{1}{4}} \quad \text{(xii)} \quad 27^{\frac{1}{3}} \times 4^{\frac{1}{2}} \quad \text{(xiii)} \quad 8^{\frac{2}{3}} \times 4^{\frac{3}{2}}$$

$$\text{(xiv)} \quad \frac{125^{\frac{2}{3}}}{16^{\frac{3}{4}}} \quad \text{(xv)} \quad \frac{49^{\frac{1}{2}} \times 8^{\frac{2}{3}}}{4^{\frac{3}{2}}}$$

$$\text{(xvi)} \quad \left(9^{\frac{3}{2}}\right)^2 \quad \text{(xvii)} \quad 4^{\frac{1}{2}} \times 8^{\frac{1}{3}} \quad \text{(xviii)} \quad 25^{-\frac{3}{2}} \quad \text{(xix)} \quad 16^{\frac{1}{4}} \times 16^{\frac{1}{2}}$$

$$\text{(xx)} \quad \frac{81^{\frac{3}{4}}}{81^{\frac{1}{4}}} \quad \text{(xxi)} \quad \left(4^{\frac{1}{2}} \times 125^{\frac{1}{3}}\right)^2 \quad \text{(xxii)} \quad \left(\frac{9^{\frac{1}{2}}}{4^{\frac{1}{2}}}\right)^2$$

2. Write the following numbers using radical signs and find their values. Also find the values of the same, assuming the first law of indices to be true for rational indices. Verify that you get the same answers. Study the solved example.

$$\begin{aligned} \text{e.g. (i)} \quad 16^{\frac{1}{4}} \times 16^{\frac{3}{4}} &= \sqrt[4]{16} \times \sqrt[4]{16^3} \\ &= 2 \times \sqrt[4]{4196} \\ &= 2 \times 8 \\ &= 16 \end{aligned} \quad \begin{aligned} \text{(ii)} \quad 16^{\frac{1}{4}} \times 16^{\frac{3}{4}} &= 16^{\frac{1}{4} + \frac{3}{4}} \\ &= 16^1 \\ &= 16 \end{aligned}$$

$$\text{(i)} \quad 81^{\frac{1}{2}} \times 81^{\frac{1}{4}}$$

$$\text{(ii)} \quad 32^{\frac{2}{5}} \times 32^{\frac{1}{5}} \times 32^{\frac{2}{5}}$$

$$\text{(iii)} \quad 27^{\frac{2}{3}} \times 27^{\frac{1}{3}}$$

$$\text{(iv)} \quad 8^{\frac{1}{3}} \times 8^{\frac{2}{3}} \times 8^{\frac{1}{3}}$$

$$\text{(v)} \quad 27^{\frac{2}{3}} \times 27^{\frac{1}{3}}$$

§ 5.4 $a^{-r} = \frac{1}{a^r}$, when r is a rational number :

You know that $a^{-r} = \frac{1}{a^r}$, when r is an integer. Is this property true even when r is any rational number? We have already assumed that the first law of indices is true even for rational indices. Using this, let us establish that $a^{-r} = \frac{1}{a^r}$ is true for rational values of r . For this, study the following example.

$$\begin{aligned} 5^{\frac{2}{3}} \times 5^{-\frac{2}{3}} &= 5^{\frac{2}{3} + (-\frac{2}{3})} && \dots (a^m \times a^n = a^{m+n} \text{ for rational} \\ &= 5^0 && \text{values of } m \text{ and } n) \\ &= 1 && \dots (a^0 = 1) \\ \therefore 5^{\frac{2}{3}} \times 5^{-\frac{2}{3}} &= 1 && \dots (1) \end{aligned}$$

Dividing both sides of (1) by $5^{\frac{2}{3}}$, we get

$$\frac{5^{\frac{2}{3}} \times 5^{-\frac{2}{3}}}{5^{\frac{2}{3}}} = \frac{1}{5^{\frac{2}{3}}} \therefore 5^{-\frac{2}{3}} = \frac{1}{5^{\frac{2}{3}}} \dots (2)$$

Similarly dividing both sides of (1) by $5^{-\frac{2}{3}}$ we get

$$\frac{5^{\frac{2}{3}} \times 5^{-\frac{2}{3}}}{5^{-\frac{2}{3}}} = \frac{1}{5^{-\frac{2}{3}}} \therefore 5^{\frac{2}{3}} = \frac{1}{5^{-\frac{2}{3}}} \dots (3)$$

From (2) and (3) above, we get

$$5^{-\frac{2}{3}} = \frac{1}{5^{\frac{2}{3}}} \text{ and } 5^{\frac{2}{3}} = \frac{1}{5^{-\frac{2}{3}}}$$

As $5^{\frac{2}{3}} \times 5^{-\frac{2}{3}} = 1$, we see that $5^{\frac{2}{3}}$ and $5^{-\frac{2}{3}}$ are multiplicative inverses of each other.

Meaning of $x^{-p/q}$, p and q are natural numbers.

The procedure used in assigning meaning to $5^{-\frac{2}{3}}$ enables us to assign meaning to $x^{-p/q}$.

$$\begin{aligned} x^{p/q} \times x^{-p/q} &= x^{[p/q + (-p/q)]} && \dots (a^m \times a^n = a^{m+n} \text{ for rational} \\ &= x^0 && \text{values of } m \text{ and } n) \\ &= 1 && \dots (\because a^0 = 1) \\ \therefore x^{p/q} \times x^{-p/q} &= 1 && \dots (1) \end{aligned}$$

Dividing both sides of (1) by $x^{p/q}$, we get

$$\begin{aligned} \frac{x^{p/q} \times x^{-p/q}}{x^{p/q}} &= \frac{1}{x^{p/q}} \\ \therefore x^{-p/q} &= \frac{1}{x^{p/q}} && \dots (2) \end{aligned}$$

Similarly dividing both sides of (1) by $x^{-p/q}$, we get

$$\frac{x^{p/q} \times x^{-p/q}}{x^{-p/q}} = \frac{1}{x^{-p/q}}$$

$$\therefore x^{p/q} = \frac{1}{x^{-p/q}} \quad \dots (3)$$

From (2) and (3) above we get

$$x^{-p/q} = \frac{1}{x^{p/q}} \text{ and } x^{p/q} = \frac{1}{x^{-p/q}}$$

As $x^{p/q} \times x^{-p/q} = 1$, it is obvious that $x^{p/q}$ and $x^{-p/q}$ are multiplicative inverses of each other.

From the examples given above, it is clear that the property $a^{-r} = \frac{1}{a^r}$ or $a^{-r} = \frac{1}{a^r}$ which is true for integers, holds good even if the index is any rational number.

With the help of these results it is possible to convert the numbers having negative indices into numbers having positive indices and vice versa.

Let us work out some examples using these results.

$$\begin{aligned} \text{(i)} \quad 4^{-\frac{3}{2}} &= \frac{1}{4^{\frac{3}{2}}} & \text{(ii)} \quad (27)^{-\frac{2}{3}} &= \frac{1}{(27)^{\frac{2}{3}}} & \text{(iii)} \quad 4^{-\frac{3}{8}} &= \frac{1}{4^{\frac{3}{8}}} \\ &= \frac{1}{\sqrt[4]{4^3}} & &= \frac{1}{\sqrt[3]{(27)^2}} & &= \frac{1}{\sqrt[5]{4^3}} \\ &= \frac{1}{\sqrt[4]{64}} & &= \frac{1}{\sqrt[3]{729}} & &= \frac{1}{\sqrt[5]{64}} \\ &= \frac{1}{8} & &= \frac{1}{9} & & \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad (x^3)^{-\frac{2}{3}} &= \frac{1}{(x^3)^{\frac{2}{3}}} & \text{(v)} \quad (a^2 \times b^4)^{\frac{3}{2}} &= \sqrt{(a^2 \times b^4)^3} & \text{(vi)} \quad \left(\frac{x^2}{y^5}\right)^{\frac{3}{2}} &= \sqrt[3]{\left(\frac{x^2}{y^5}\right)^2} \\ &= \frac{1}{\sqrt[3]{(x^3)^2}} & &= \sqrt{a^6 \times b^{12}} & &= \sqrt[3]{\frac{x^4}{y^{10}}} \\ &= \frac{1}{\sqrt[3]{x^6}} & &= a^3 b^6 & & \\ &= \frac{1}{x^2} & & & & \end{aligned}$$

$$\begin{aligned} \text{(vii)} \quad \left(\frac{a^8}{x^4}\right)^{-\frac{2}{3}} &= \sqrt[3]{\left(\frac{a^8}{x^4}\right)^{-2}} & \text{(viii)} \quad (x^4)^{\frac{3}{4}} &= \sqrt[4]{(x^4)^3} \\ &= \sqrt[3]{\frac{a^{-16}}{x^{-8}}} & &= \sqrt[4]{x^{12}} \\ &= \sqrt[3]{\frac{x^8}{a^{16}}} & &= x^3 \end{aligned}$$

EXERCISE 5.4

1. Express the following numbers in the form of positive indices and then find their values.

(i) $8^{-\frac{2}{3}}$ (ii) $(\frac{1}{125})^{-\frac{1}{3}}$ (iii) $9^{-\frac{2}{3}}$ (iv) $4^{-\frac{3}{2}}$ (v) $(\frac{1}{8})^{-\frac{2}{3}}$ (vi) $(81)^{-\frac{3}{4}}$
 (vii) $4^{-\frac{3}{2}}$ (viii) $5^{-\frac{3}{4}}$ (ix) $(10)^{-\frac{3}{8}}$ (x) $9^{-\frac{3}{2}}$ (xi) $\frac{1}{9^{-\frac{3}{2}}}$ (xii) $\frac{1}{x^{-\frac{7}{8}}}$
 (xiii) $\frac{1}{(xy)^{-\frac{4}{7}}}$ (xiv) $\frac{1}{(49)^{-\frac{3}{2}}}$ (xv) $\frac{1}{a^{-\frac{7}{8}}}$ (xvi) $\frac{x^{-\frac{2}{8}}}{x^{-\frac{3}{4}}}$
 (xvii) $a^{-\frac{4}{3}} b^{-\frac{5}{4}}$ (xviii) $\frac{1}{x^{-3} b^{-5}}$ (xix) $x^{-\frac{2}{3}} + y^{-\frac{2}{3}} = a^{-\frac{2}{3}}$
 (xx) $(x+y)^{-1} (y+z)^{-2}$.

2. Express the following numbers in the form of the numbers consisting of negative indices.

(i) $(13)^{\frac{2}{3}}$ (ii) $\frac{1}{5^{\frac{2}{7}}}$ (iii) $\frac{1}{a^{\frac{3}{4}} b^{\frac{2}{3}}}$ (iv) $x^{\frac{4}{7}}$ (v) $x^{\frac{3}{2}} y^{\frac{4}{7}}$ (vi) $\frac{2^x}{3^y}$
 (vii) $(\frac{2}{3})^{\frac{2}{7}}$ (viii) $\frac{1}{(\frac{5}{7})^{\frac{3}{8}}}$ (ix) $(\frac{1}{3})^{\frac{4}{5}}$ (x) $(x-1)^{\frac{4}{5}}$.

3. Simplify.

(i) $2^{17} \times 4^{\frac{3}{2}}$ (ii) $3^{15} \div (27)^{\frac{2}{3}}$ (iii) $2^{13} \times 8^{-\frac{2}{3}}$ (iv) $(25)^{-\frac{3}{2}} \div 5^{12}$
 (v) $2^{-7} \times 8^{\frac{2}{3}}$ (vi) $2^{-9} \times (16)^{-\frac{3}{4}}$ (vii) $9^{\frac{1}{2}} \div (27)^{-\frac{2}{3}}$
 (viii) $(25)^{-\frac{5}{2}} \div (125)^{\frac{2}{3}}$ (ix) $(x^2)^{\frac{3}{2}}$ (x) $(a^3)^{-\frac{4}{5}}$ (xi) $(8^{\frac{2}{3}})^7$
 (xii) $[(18)^{-\frac{3}{4}}]^{-3}$ (xiii) $(b^4)^{\frac{3}{2}}$ (xiv) $(x^3 y^6)^{-\frac{3}{2}}$ (xv) $(m^{-4} n^{-8})^{-\frac{3}{2}}$
 (xvi) $(\frac{a^2}{b^4})^{\frac{3}{2}}$ (xvii) $(\frac{x^4}{y^{-8}})^{-\frac{3}{2}}$ (xviii) $(\frac{a^{-3}}{b^{-6}})^{-\frac{1}{2}}$ (xix) $(\frac{x^{-10}}{y^5})^{\frac{2}{5}}$
 (xx) $(\frac{2^{-6}}{3^{-9}})^{-\frac{2}{3}}$.

4. Assuming that $a^m \times a^n \times a^p \times \dots = a^{m+n+p+\dots}$ to be true for all rational values of $m, n, p, \dots$, Simplify.

(i) $2^{\frac{3}{2}} \times 2^0 \times 2^{-1}$ (ii) $5^{-\frac{1}{2}} \times 5^{-3} \times 5^4$ (iii) $x^{\frac{4}{3}} \div x^{\frac{2}{3}}$ (iv) $b^{\frac{1}{2}} \div b^{-\frac{3}{2}}$
 (v) $a^{-\frac{1}{3}} \div a^{\frac{3}{4}}$ (vi) $x^{-\frac{2}{3}} \times x^{-\frac{1}{4}} \times x^0$ (vii) $x^{-4} \times x^{-\frac{3}{2}} \times x^0 \times x^2$.

§ 5.5 Laws of Indices :

Assuming the first law of indices to be true even for rational indices, we have assigned meanings to the numbers having rational indices. With the help of this meaning, we shall verify from some examples that the other laws of indices are also true when the index is a rational number.

(I) $a^m \times a^n \times a^p \times \dots = a^{m+n+p+\dots}$ for all rational values of $m, n, p, \dots$. This law we have already assumed.

§ 5.6 (II) $a^m \div a^n = a^{m-n}$, for all rational values of m and n :

$$\begin{aligned} \text{(i) } a^{\frac{5}{2}} \div a^{\frac{3}{4}} &= \frac{a^{\frac{5}{2}}}{a^{\frac{3}{4}}} = \frac{a^{\frac{5}{2}}}{1} \times \frac{1}{a^{\frac{3}{4}}} \quad \dots \left(\because \frac{x}{y} = x \times \frac{1}{y} \right) \\ &= a^{\frac{5}{2}} \times a^{-\frac{3}{4}} \quad \dots \left(\because \frac{1}{a^r} = a^{-r} \right) \\ &= a^{\frac{5}{2} + (-\frac{3}{4})} \quad \dots (a^m \times a^n = a^{m+n} \text{ for rational values of } m \text{ and } n) \\ &= a^{\frac{7}{4}} \quad \dots (1) \end{aligned}$$

$$\begin{aligned} a^{\frac{5}{2}} \div a^{\frac{3}{4}} &= \frac{a^{\frac{5}{2}}}{a^{\frac{3}{4}}} = a^{\frac{5}{2} - \frac{3}{4}} \\ &= a^{\frac{7}{4}} \quad \dots (2) \quad \left(\text{Using } a^m \div a^n = a^{m-n} \right) \end{aligned}$$

From (1) and (2), we have $a^m \div a^n = a^{m-n}$ is true when m and n are positive fractions.

Similarly verify that the property $a^m \div a^n = a^{m-n}$ is true for different rational values of m and n .

(ii) We shall now verify that $a^m \div a^n = a^{m-n}$ is true for all rational values of m and n .

$$\begin{aligned} a^m \div a^n &= \frac{a^m}{a^n} \\ &= \frac{a^m}{1} \times \frac{1}{a^n} \quad \dots \left(\because \frac{x}{y} = x \times \frac{1}{y} \right) \\ &= a^m \times a^{-n} \quad \dots \left(\because \frac{1}{a^r} = a^{-r} \right) \\ &= a^{m + (-n)} \quad \dots (a^m \times a^n = a^{m+n} \text{ for rational } m \text{ and } n) \\ &= a^{m-n} \end{aligned}$$

$\therefore a^m \div a^n = a^{m-n}$, for all rational values of m and n .

Using the method mentioned above, verify that the law $a^m \div a^n = a^{m-n}$ is true for rational values of m and n , for the examples given below.

- | | | | |
|---------------------------------|---|---|--|
| (i) $a^5 \div a^{\frac{3}{2}}$ | (ii) $x^7 \div x^{-\frac{3}{4}}$ | (iii) $2^{-3} \div 2^{\frac{7}{2}}$ | (iv) $5^{-4} \div 5^{-\frac{5}{2}}$ |
| (v) $x^{\frac{2}{3}} \div x^4$ | (vi) $m^{-\frac{2}{3}} \div m^{-\frac{3}{2}}$ | (vii) $b^{-\frac{4}{3}} \div b^{\frac{2}{5}}$ | (viii) $x^{\frac{2}{3}} \div x^{-\frac{1}{3}}$ |
| (ix) $a^0 \div a^{\frac{2}{3}}$ | (x) $y^{-\frac{2}{3}} \div y^0$ | | |

Study the following examples based on the first two laws of indices.

Simplify.

$$(i) x^{-\frac{7}{2}} \div x^{\frac{1}{2}} \quad (ii) \frac{x^{-\frac{5}{2}} \times x^{\frac{3}{2}}}{x^{\frac{5}{4}}} \quad (iii) \frac{a^{\frac{2}{3}} b^{-\frac{7}{2}} \times b^{\frac{2}{3}} c^{-\frac{1}{3}}}{b^{\frac{2}{3}} c^{\frac{3}{4}} \times c^{\frac{1}{2}} b^{-2}}$$

Solution

$$(i) x^{-\frac{7}{2}} \div x^{\frac{1}{2}} = x^{-\frac{7}{2} - \frac{1}{2}}$$

$$= x^{-\frac{23}{6}} = \frac{1}{x^{\frac{23}{6}}}$$

$$(ii) \frac{x^{-\frac{5}{2}} \times x^{\frac{3}{2}}}{x^{\frac{5}{4}}} = x^{-\frac{5}{2} + \frac{3}{2} - \frac{5}{4}}$$

$$= x^{-\frac{17}{4}} = \frac{1}{x^{\frac{17}{4}}}$$

$$(iii) \frac{a^{\frac{2}{3}} b^{-\frac{7}{2}} \times b^{\frac{2}{3}} c^{-\frac{1}{3}}}{b^{\frac{2}{3}} c^{\frac{3}{4}} \times c^{\frac{1}{2}} b^{-2}} = \frac{a^{\frac{2}{3}} b^{-\frac{7}{2} + \frac{2}{3}} c^{-\frac{1}{3}}}{b^{\frac{2}{3} + (-2)} c^{\frac{3}{4} + \frac{1}{2}}} \\ = a^{\frac{2}{3}} b^{-\frac{7}{2} + \frac{2}{3} - (\frac{3}{2} - 2)} c^{-\frac{1}{3} - (\frac{3}{4} + \frac{1}{2})} \\ = a^{\frac{2}{3}} b^{-\frac{37}{6}} c^{-\frac{19}{4}} = \frac{a^{\frac{2}{3}}}{b^{\frac{37}{6}} c^{\frac{19}{4}}}$$

EXERCISE 5.5

1. Simplify.

$$(i) \frac{2^{\frac{3}{2}}}{2^{\frac{1}{2}}}$$

$$(ii) 5^{-\frac{1}{3}} \div 5^2$$

$$(iii) \frac{x^{\frac{1}{3}} y^{-\frac{2}{3}}}{x^{-\frac{1}{6}} y^{\frac{3}{2}}}$$

$$(iv) \sqrt[4]{a^5} \div \sqrt[4]{a^{-3}}$$

$$(v) \frac{\sqrt[3]{x^2} \times \sqrt[4]{x^3} \times \sqrt{x}}{\sqrt{x^3} \times \sqrt[4]{x^2}}$$

$$(vi) \frac{x^{\frac{2}{3}} y^{-\frac{1}{3}} \times y^{-\frac{2}{3}} z^{\frac{1}{3}} \times z^{-\frac{1}{2}} x^{\frac{2}{3}}}{x^{\frac{1}{2}} y^{-\frac{2}{3}} z^{\frac{1}{3}} \times z^2 x^{-\frac{3}{2}} y^{\frac{2}{3}}}$$

$$(vii) \frac{x^{-1} + x^{-\frac{1}{2}} + x^0 + x^{\frac{1}{2}}}{x^{\frac{3}{2}}}$$

2. Expand.

$$(i) x^{\frac{3}{2}} (x^{-\frac{1}{2}} + x^{-1} + x^{\frac{3}{2}})$$

$$(ii) a^{-\frac{1}{2}} (a^{\frac{2}{3}} - 2a^{-\frac{3}{2}})$$

$$(iii) a^{\frac{1}{3}} b^{-\frac{2}{3}} (a^{-\frac{1}{2}} b^{\frac{1}{2}} - a^{\frac{1}{2}} b^{\frac{3}{2}} + a^{-\frac{2}{3}} b^{-\frac{1}{3}})$$

$$(iv) (x^{\frac{1}{2}} + y^{\frac{1}{3}}) (x^{-\frac{1}{2}} - y^{\frac{2}{3}}) \quad (v) (2a - b^{\frac{1}{3}}) (a^{-\frac{1}{3}} + 3b)$$

$$(vi) (x + y^{\frac{1}{2}})^2 \quad (vii) (x^{\frac{1}{2}} - y^{\frac{1}{3}})^3 \quad (viii) (x^{\frac{1}{2}} + x^{-\frac{1}{2}})^2$$

§ 5.7 (III) $(a^m)^n = a^{mn}$, m and n are rational numbers :

Let us verify this law by solving some examples.

$$\begin{aligned}
 \text{(i)} \quad \left(a^{\frac{2}{3}}\right)^{\frac{5}{7}} &= \sqrt[7]{\left(a^{\frac{2}{3}}\right)^5} && \dots \left(a^{p/q} = \sqrt[q]{a^p}\right) \\
 &= \sqrt[7]{a^{\frac{2}{3} \times 5}} && \dots \text{(Definition of } a^m) \\
 &= \sqrt[7]{a^{\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3}}} && \dots (a^m \times a^n = a^{m+n} \text{ for rational values of } m \text{ and } n) \\
 &= \sqrt[7]{a^{\frac{2}{3} \times 5}} = \sqrt[7]{a^{\frac{10}{3}}} \\
 &= a^{(\frac{10}{3})/7} && \dots \left(\sqrt[q]{a^p} = a^{p/q}\right) \\
 &= a^{\frac{10}{21}} && \dots (1) \\
 a^{\frac{2}{3} \times \frac{5}{7}} &= a^{\frac{10}{21}} && \dots (2)
 \end{aligned}$$

From (1) and (2) we get $\left(a^{\frac{2}{3}}\right)^{\frac{5}{7}} = a^{\frac{2}{3} \times \frac{5}{7}}$

Therefore, when m and n are positive rational numbers $(a^m)^n = a^{mn}$ is true.

$$\begin{aligned}
 \text{(ii)} \quad \left(x^{\frac{3}{5}}\right)^{-\frac{2}{3}} &= \frac{1}{\left(x^{\frac{3}{5}}\right)^{\frac{2}{3}}} && \dots \left(\because a^{-r} = \frac{1}{a^r}\right) \\
 &= \frac{1}{\sqrt[3]{\left(x^{\frac{3}{5}}\right)^2}} && \dots \left(\because a^{p/q} = \sqrt[q]{a^p}\right) \\
 &= \frac{1}{\sqrt[3]{x^{\frac{3}{5} \times 2}}} && \dots \text{(Definition of } a^m) \\
 &= \frac{1}{\sqrt[3]{x^{\frac{3}{5} + \frac{3}{5}}}} && \dots (a^m \times a^n = a^{m+n} \text{ for all rational values of } m \text{ and } n) \\
 &= \frac{1}{\sqrt[3]{x^{\frac{6}{5}}}} = \frac{1}{x^{(\frac{6}{5})/3}} && \dots \left(\sqrt[q]{a^p} = a^{p/q}\right) \\
 &= \frac{1}{x^{\frac{2}{5}}} = x^{-\frac{2}{5}} && \dots (1) \quad \left(\because \frac{1}{a^r} = a^{-r}\right) \\
 x^{\frac{3}{5} \times (-\frac{2}{3})} &= x^{-\frac{2}{5}} && \dots (2)
 \end{aligned}$$

$\therefore$ From (1) and (2) we have $\left(x^{\frac{3}{5}}\right)^{-\frac{2}{3}} = x^{\frac{3}{5} \times (-\frac{2}{3})}$ Hence $(a^m)^n = a^{mn}$ is true for m a positive and n a negative rational.

Similarly solving following examples, verify that the law $(a^m)^n = a^{mn}$ is true when m and n are rational numbers.

Now, study the following examples based on this property.

Simplify. (i) $(2^{\frac{3}{2}})^{-2} \times (2^{\frac{2}{5}})^{\frac{5}{3}} \times (2^{-\frac{1}{3}})^{\frac{2}{5}}$ (ii) $\left[(x^{\frac{2}{3}})^{\frac{4}{5}}\right]^{\frac{3}{2}}$ (iii) $x^{(\frac{4}{9})^{\frac{3}{2}}}$

(iv) $(x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})$ (v) $\frac{(a^{\frac{2}{3}})^{-\frac{1}{3}} \times (b^{-\frac{3}{5}})^2 \times (a^{\frac{1}{2}})^3}{(b^{\frac{2}{3}})^{-2} \times (a^{-\frac{1}{3}})^4 \times (b^{\frac{4}{5}})^{-\frac{2}{3}}}$

(vi) $(256)^{\frac{3}{8}}$

Solution :

(i) $(2^{\frac{3}{2}})^{-2} \times (2^{\frac{2}{5}})^{\frac{5}{3}} \times (2^{-\frac{1}{3}})^{\frac{2}{5}} = 2^{-3} \times 2^{\frac{2}{3}} \times 2^{-\frac{2}{15}} = 2^{-3 + \frac{2}{3} + (-\frac{2}{15})} = 2^{-\frac{17}{15}} = \frac{1}{2^{\frac{17}{15}}}$

(ii) $\left[(x^{\frac{2}{3}})^{\frac{4}{5}}\right]^{\frac{3}{2}} = [x^{\frac{8}{15}}]^{-\frac{3}{2}} = x^{-\frac{4}{5}} = \frac{1}{x^{\frac{4}{5}}}$

(iii) $x^{(\frac{4}{9})^{\frac{3}{2}}} = x^{\sqrt{(\frac{4}{9})^3}} = x^{\sqrt{\frac{64}{27}}} = x^{\frac{8}{9}}$

(iv) $(x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})$
 $= (x^{\frac{1}{3}} - y^{\frac{1}{3}}) \left[(x^{\frac{1}{3}})^2 + x^{\frac{1}{3}}y^{\frac{1}{3}} + (y^{\frac{1}{3}})^2 \right]$
 $= (x^{\frac{1}{3}})^3 - (y^{\frac{1}{3}})^3 \quad [\because (a-b)(a^2+ab+b^2) = a^3 - b^3]$
 $= x - y$

(v) $\frac{(a^{\frac{2}{3}})^{-\frac{1}{3}} \times (b^{-\frac{3}{5}})^2 \times (a^{\frac{1}{2}})^3}{(b^{\frac{2}{3}})^{-2} \times (a^{-\frac{1}{3}})^4 \times (b^{\frac{4}{5}})^{-\frac{2}{3}}} = \frac{a^{-\frac{2}{9}} \times b^{-\frac{6}{5}} \times a^{\frac{3}{2}}}{b^{\frac{4}{3}} \times a^{-\frac{4}{3}} \times b^{-\frac{8}{15}}}$
 $= a^{-\frac{2}{9} + \frac{3}{2} - (-\frac{4}{3})} \times b^{-\frac{6}{5} - \frac{4}{3} - (-\frac{8}{15})}$
 $= a^{\frac{17}{6}} \times b^{-\frac{8}{5}} = \frac{a^{\frac{17}{6}}}{b^{\frac{8}{5}}}$

(vi) $(256)^{\frac{3}{8}} = \sqrt[8]{(256)^3}$. To solve this, we first find the cube of 256 and then find the 8th root. This certainly involves laborious calculations. To reduce this labour, we shall try to express $a^{p/q} = \sqrt[q]{a^p}$ in another form.

For all rational values of m and n , we have $(a^m)^n = a^{mn}$ and $(a^n)^m = a^{nm} = a^{mn}$.
 $\therefore (a^m)^n = (a^n)^m$.

Let us use this form of the law to solve the given example.

Instead of $a^{p/q} = (a^p)^{1/q}$, if we use the form $a^{p/q} = (a^{1/q})^p$, it helps us to evaluate the given number. Because, comparatively it is easier to find the root first

and then to raise it to a power. $\therefore (256)^{\frac{3}{8}} = \left[(256)^{\frac{1}{8}}\right]^3 = (2)^3 = 8$.

2. Expand. $(x^{\frac{1}{6}} - y^{\frac{1}{6}})^3$.

$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$. Using this

$$\begin{aligned}(x^{\frac{1}{6}} - y^{\frac{1}{6}})^3 &= (x^{\frac{1}{6}})^3 - 3(x^{\frac{1}{6}})^2 y^{\frac{1}{6}} + 3x^{\frac{1}{6}} (y^{\frac{1}{6}})^2 - (y^{\frac{1}{6}})^3 \\ &= x^{\frac{1}{2}} - 3x^{\frac{1}{3}} y^{\frac{1}{6}} + 3x^{\frac{1}{6}} y^{\frac{1}{2}} - y^{\frac{1}{2}}\end{aligned}$$

3. Factorise $x^{\frac{1}{4}} - y^{\frac{1}{4}}$ by expressing it in the form $a^2 - b^2$.

$$\begin{aligned}x^{\frac{1}{4}} - y^{\frac{1}{4}} &= (x^{\frac{1}{8}})^2 - (y^{\frac{1}{8}})^2 \\ &= (x^{\frac{1}{8}} - y^{\frac{1}{8}})(x^{\frac{1}{8}} + y^{\frac{1}{8}})\end{aligned}$$

4. Factorise $x^{\frac{1}{2}} - y^{\frac{1}{2}}$ by expressing it in the form $a^3 - b^3$.

$$\begin{aligned}x^{\frac{1}{2}} - y^{\frac{1}{2}} &= (x^{\frac{1}{6}})^3 - (y^{\frac{1}{6}})^3 \\ &= (x^{\frac{1}{6}} - y^{\frac{1}{6}}) \left[(x^{\frac{1}{6}})^2 + x^{\frac{1}{6}} y^{\frac{1}{6}} + (y^{\frac{1}{6}})^2 \right] \\ &= (x^{\frac{1}{6}} - y^{\frac{1}{6}}) (x^{\frac{1}{3}} + x^{\frac{1}{6}} y^{\frac{1}{6}} + y^{\frac{1}{3}})\end{aligned}$$

5. Express the following numbers having indices with smallest common denominator, $2^{\frac{1}{4}}$, $3^{\frac{1}{3}}$ and $5^{\frac{1}{2}}$.

Here $\frac{1}{4}$, $\frac{1}{3}$ and $\frac{1}{2}$ are the indices of the given numbers. Least common multiple (L. C. M.) of 4, 3, and 2 is 12. Hence, it is necessary to express every number with $\frac{1}{12}$ as the index.

(i) $2^{\frac{1}{4}} = (2^3)^{\frac{1}{12}} = 3^{\frac{1}{12}}$ (ii) $3^{\frac{1}{3}} = (3^4)^{\frac{1}{12}} = (31)^{\frac{1}{12}}$

(iii) $5^{\frac{1}{2}} = (5^6)^{\frac{1}{12}} = (15625)^{\frac{1}{12}}$.

EXERCISE 5.6

1. Simplify :

(i) $(5^{\frac{1}{3}})^2 \times (5^{-\frac{3}{2}})^{-4} \times (5^{\frac{4}{3}})^{\frac{3}{2}}$ (ii) $[(a^{-\frac{1}{3}})^{\frac{4}{5}}]^{\frac{3}{5}}$ (iii) $a^{(\frac{3}{27})^{\frac{2}{3}}}$

(iv) $(x^2)^{-\frac{2}{3}}$ (v) $(a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{2}{3}} - a^{\frac{1}{3}} b^{\frac{1}{3}} + b^{\frac{2}{3}})$ (vi) $(243)^{\frac{2}{3}}$

(vii) $(625)^{\frac{3}{4}}$ (viii) $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{1}{6}} + y^{\frac{1}{6}})(x^{\frac{1}{12}} + y^{\frac{1}{12}})(x^{\frac{1}{12}} - y^{\frac{1}{12}})$

(ix) $(32)^{\frac{1}{5}}$ (x) $3^1 \times 9^{\frac{1}{2}} \times (27)^{\frac{1}{3}} \times (81)^{\frac{1}{4}}$ (xi) $5^{\frac{1}{3}} \times (25)^{\frac{3}{2}} \times (125)^{\frac{1}{3}}$

(xii) $8^{\frac{2}{3}} \times (27)^{\frac{4}{3}} \times 9^{\frac{2}{3}} \times 4^{\frac{3}{4}} \times (64)^{\frac{1}{4}}$ (xiii) $[(x^{\frac{3}{5}})^0]^{-\frac{7}{2}}$

(xiv) $\frac{(x^{\frac{1}{2}})^3 \times (y^2)^{-\frac{3}{2}} \times (z^{\frac{1}{3}})^{-4} \times (y^{-\frac{2}{3}})^2 \times (x^{-\frac{1}{3}})^{-\frac{3}{2}}}{(y^{\frac{1}{2}})^{\frac{4}{3}} \times (x^{\frac{1}{3}})^2 \times (z^{-\frac{3}{4}})^{-4} \times (x^{\frac{3}{5}})^{-\frac{1}{2}}}$ (xv) $\frac{(125)^{\frac{2}{3}}}{(16)^{\frac{3}{4}}}$

$$(xvi) \frac{(49)^{\frac{1}{2}} \times 8^{\frac{2}{3}}}{4^{\frac{3}{2}}} \quad (xvii) [(x+y+z)^{\frac{1}{3}}]^3 \times [(x+y+z)^4]^{-\frac{1}{4}}$$

2. Expand.

$$(i) (x^{\frac{1}{2}} + y^{\frac{1}{2}})^2 \quad (ii) (x^{\frac{2}{3}} - x^{-\frac{2}{3}})^2 \quad (iii) (x^{\frac{4}{3}} + y^{\frac{4}{3}})^3$$

$$(iv) (x^{\frac{1}{3}} + x^{-\frac{1}{3}})^3 \quad (v) (x^{\frac{1}{3}} y^{-\frac{1}{3}} - x^{-\frac{1}{3}} y^{\frac{1}{3}})^3$$

3. Express the following expressions in the form of perfect square or cube as the case may be.

$$(i) x^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} \quad (ii) x^{\frac{1}{2}} + 3x^{\frac{1}{3}}y^{\frac{1}{6}} + 3x^{\frac{1}{6}}y^{\frac{1}{3}} + y^{\frac{1}{2}}$$

$$(iii) a^{\frac{1}{2}} - 2 + a^{-\frac{1}{2}} \quad (iv) x - 3(x^{\frac{1}{3}} - x^{-\frac{1}{3}}) - x^{-1} \quad (v) a + 2 + a^{-1}$$

4. Express in the form of $a^2 - b^2$ and factorise.

$$(i) x - y \quad (ii) x^{\frac{1}{2}} - y^{\frac{1}{2}} \quad (iii) x^{\frac{3}{5}} - y^{\frac{3}{5}} \quad (iv) x^{-1} - y^{-1}$$

5. Express in the form $a^3 - b^3$ or $a^3 + b^3$ and factorise.

$$(i) x - y \quad (ii) x^{\frac{1}{2}} + y^{\frac{1}{2}} \quad (iii) x^{\frac{1}{3}} + y^{\frac{1}{3}} \quad (iv) x^{\frac{1}{4}} - y^{\frac{1}{4}}$$

6. Express in the form of equal indices.

$$(i) 3^{\frac{1}{2}}, 2^{\frac{1}{3}}, 5^{\frac{1}{5}} \quad (ii) 3^{\frac{1}{3}}, 5^{\frac{2}{3}}, 7^{\frac{2}{3}} \quad (iii) x^{\frac{1}{2}}, x^{\frac{2}{3}}, x^{\frac{1}{4}}$$

§ 5.8 (IV) $(a \times b)^m = a^m \times b^m$, for all rational values of m :

We shall now verify that $(a \times b)^m = a^m \times b^m$ for all rational values of m . For this study the following examples.

$$(i) [(a \times b)^{-\frac{2}{3}}]^3 = (a \times b)^{-\frac{2}{3} \times 3} \quad \dots \left[(a^m)^n = a^{mn} \text{ for all rational values.} \right]$$

$$= (a \times b)^{-2} = a^{-2} \times b^{-2} \quad \dots \left[(a \times b)^m = a^m \times b^m \text{ for integers} \right]$$

$$\therefore [(a \times b)^{-\frac{2}{3}}]^3 = a^{-2} \times b^{-2}$$

Taking cube root of both sides, we get

$$(a \times b)^{-\frac{2}{3}} = \sqrt[3]{a^{-2} \times b^{-2}} \quad \dots (1)$$

$$(a^{-\frac{2}{3}} \times b^{-\frac{2}{3}})^3 = (a^{-\frac{2}{3}})^3 \times (b^{-\frac{2}{3}})^3 \quad \dots \left[(ab)^m = a^m \times b^m \right]$$

$$= a^{-\frac{2}{3} \times 3} \times b^{-\frac{2}{3} \times 3} \quad \dots \left[(a^m)^n = a^{mn} \text{ for rational values.} \right]$$

$$= a^{-2} \times b^{-2}$$

$$\therefore (a^{-\frac{2}{3}} \times b^{-\frac{2}{3}})^3 = a^{-2} \times b^{-2}$$

Taking cube root of both sides, we get

$$a^{-\frac{2}{3}} \times b^{-\frac{2}{3}} = \sqrt[3]{a^{-2} \times b^{-2}} \quad \dots (2)$$

From (1) and (2), we have $(a \times b)^{-\frac{2}{3}} = a^{-\frac{2}{3}} \times b^{-\frac{2}{3}}$

Hence, $(a \times b)^m = a^m \times b^m$ is true when m is a negative rational number.

You know that $(a \times b)^m = a^m \times b^m$ when m is a positive integer, zero or a negative integer. You have verified above that $(a \times b)^m = a^m \times b^m$ is true for negative rational values of m . Similarly, you can verify this property for positive rational values by solving the following examples.

$$(i) (x \times y)^{\frac{5}{4}} \quad (ii) (abc)^{\frac{3}{5}} \quad (iii) (xyz)^{\frac{7}{4}}$$

From the above examples, it is clear that $(a \times b)^m = a^m \times b^m$ is true for all rational values of m .

§ 5.9 (V) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, m is a rational number:

Let us consider some examples to verify this law.

$$\begin{aligned} \left(\frac{x}{y}\right)^{\frac{2}{3}} &= \left(x \times \frac{1}{y}\right)^{\frac{2}{3}} && \dots \left(\because \frac{a}{b} = a \times \frac{1}{b}\right) \\ &= (x \times y^{-1})^{\frac{2}{3}} && \dots \left(\because \frac{1}{a^r} = a^{-r}\right) \\ &= x^{\frac{2}{3}} \times (y^{-1})^{\frac{2}{3}} && \dots \left[\because (a \times b)^m = a^m \times b^m \text{ for rational } m.\right] \\ &= x^{\frac{2}{3}} \times y^{-\frac{2}{3}} && \dots \left[\because (a^m)^n = a^{m \times n} \text{ for rational } m \text{ and } n.\right] \\ &= \frac{x^{\frac{2}{3}}}{y^{\frac{2}{3}}} && \dots \left(\because a^{-r} = \frac{1}{a^r}\right) \\ \therefore \left(\frac{x}{y}\right)^{\frac{2}{3}} &= \frac{x^{\frac{2}{3}}}{y^{\frac{2}{3}}} \end{aligned}$$

Hence, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ is true when m is a positive rational.

Similarly, solve the following examples and verify that $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ is true when m is a negative rational.

$$(i) \left(\frac{a}{b}\right)^{-\frac{5}{4}} \quad (ii) \left(\frac{c}{d}\right)^{-\frac{3}{5}} \quad (iii) \left(\frac{x}{y}\right)^{-\frac{1}{2}}$$

You know that $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ is true, when m is a positive integer, zero or a negative integer. We have already proved above that this law is true even when m is a positive or negative rational.

Hence, for all rational values of m , $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ holds good.

Study the example based on the laws, (i) Power of the product and (ii) Power of the quotient.

Simplify.

$$\begin{aligned} \left[\frac{a^2 b^{\frac{2}{3}}}{x^{\frac{1}{2}} y^{-2}} \right]^{-\frac{4}{5}} &= \frac{(a^2 b^{\frac{2}{3}})^{-\frac{4}{5}}}{(x^{\frac{1}{2}} y^{-2})^{-\frac{4}{5}}} \quad \dots \left[\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \right] \\ &= \frac{(a^2)^{-\frac{4}{5}} \times (b^{\frac{2}{3}})^{-\frac{4}{5}}}{(x^{\frac{1}{2}})^{-\frac{4}{5}} \times (y^{-2})^{-\frac{4}{5}}} \quad \dots \left[(a \times b)^m = a^m \times b^m \right] \\ &= \frac{a^{-\frac{8}{5}} \times b^{-\frac{8}{15}}}{x^{-\frac{2}{5}} y^{\frac{8}{5}}} = \frac{x^{\frac{2}{5}}}{a^{\frac{8}{5}} b^{\frac{8}{15}} y^{\frac{8}{5}}} \end{aligned}$$

EXERCISE 5.7

Simplify.

$$\begin{aligned} 1. (2^3 \times 5^{-\frac{2}{3}} \times 7^{\frac{4}{5}})^{-\frac{3}{7}} \quad & 2. (2^{-\frac{1}{3}} \times 5^{\frac{3}{2}} \times 7^{-2})^{\frac{2}{3}} \\ 3. \left(\frac{4x^{-3}}{y^{-\frac{2}{3}} \times z} \right)^{-\frac{3}{2}} \quad & 4. \left(\frac{a^{\frac{1}{2}} b^{-\frac{3}{2}}}{c^3 d^{\frac{1}{2}}} \right)^{-\frac{1}{3}} \quad 5. (a^{\frac{2}{3}} b^{\frac{1}{4}} \times b^{-\frac{1}{3}} c^{-\frac{1}{2}} \times a^{\frac{3}{4}} c^{\frac{1}{3}})^{-\frac{3}{4}} \\ 6. \left[\frac{a^{-\frac{2}{5}} b^{\frac{3}{4}} c^{\frac{3}{4}}}{a^{\frac{1}{2}} b^{-\frac{1}{3}} c^{-2}} \right]^{\frac{2}{3}} \quad & 7. \left[\frac{x^{\frac{1}{2}} y^{\frac{3}{2}} \times y^{-\frac{1}{3}} z^{\frac{1}{2}}}{x^{-\frac{3}{4}} y^{\frac{1}{2}} \times y^{-\frac{4}{3}} z^{\frac{1}{3}}} \right]^{-\frac{3}{5}} \quad 8. \left(\frac{\sqrt{9x^{-\frac{1}{2}}} \times \sqrt[3]{y^3}}{2x^2 z^{-1}} \right)^{-2} \end{aligned}$$

§ 5.10 Exponential Equations :

(1) It is obvious that if $x = 2$ then $x^2 = 2^2$, $x^3 = 2^3$, $x^7 = 2^7$ etc. Therefore, we can say that when equal numbers are raised to equal powers they give rise to equal numbers. i.e. If $x = y$ then $x^m = y^m$.

(2) The converse of the above statement is also true.

i.e. if $x^m = y^m$ then $x = y$, provided $m \neq 0$, and if $x > 0$, $y > 0$

Why was it necessary to impose the condition $m \neq 0$? Since any non-zero number raised to zero is equal to unity, we have $2^0 = 5^0$. To this equality, if we apply blindly the result 'If $x^m = y^m$ then $x = y$ ' we get $2 = 5$. This is certainly absurd. To avoid such absurdity, it is essential to impose the condition $m \neq 0$.

(3) If $a^x = a^y$ then $x = y$, provided $a \neq 1$. You can verify very easily that this result is true. What is the necessity of imposing the condition $a \neq 1$?

You know that any power of unity is unity. $\therefore 1^2 = 1^3$. To this equality, if we blindly apply the result 'If $a^x = a^y$ then $x = y$ ' we get $2 = 3$. This is obviously absurd and hence the necessity of the condition $a \neq 1$.

Now study, how the exponential equations can be solved using the above results.

(i) Solve. $4^{3x-1} = 8^{x+3}$

Solution : $4^{3x-1} = 8^{x+3}$

$$\therefore (2^2)^{3x-1} = (2^3)^{x+3}$$

$$\therefore 2^{6x-2} = 2^{3x+9}$$

$$\therefore 6x - 2 = 3x + 9 \dots (\because \text{If } a^x = a^y \text{ then } x = y, a \neq 1)$$

$$\therefore x = \frac{11}{3}$$

Putting $x = \frac{11}{3}$ in the given example, verify that the equation holds good.

(ii) Solve. $2^{x-y} = 4^{x-2}$ and $3^{2x-1} = (27)^{y+1}$

Solution :

$$2^{x-y} = 4^{x-2}$$

$$\therefore 2^{x-y} = (2^2)^{x-2}$$

$$\therefore 2^{x-y} = 2^{2x-4}$$

$$\therefore x - y = 2x - 4$$

$$\therefore x + y = 4 \dots \dots (1)$$

$$3^{2x-1} = (27)^{y+1}$$

$$\therefore 3^{2x-1} = (3^3)^{y+1}$$

$$\therefore 3^{2x-1} = 3^{3y+3}$$

$$\therefore 2x - 1 = 3y + 3$$

$$\therefore 2x - 3y = 4 \dots \dots (2)$$

Solving equations (1) and (2) we get $x = \frac{16}{5}$ and $y = \frac{4}{5}$

Substituting these values in the given equations verify that the equations are satisfied.

(iii) Solve. $(2x + 3)^8 (x - 1)^2 = (2x + 3)^9 (x - 1)^7$

Solution : $\therefore (2x + 3)^8 (x - 1)^2 = (2x + 3)^9 (x - 1)^7 = 0$

$$\therefore (2x + 3)^8 (x - 1)^2 [(2x + 3) - (x - 1)^5] = 0$$

$$\therefore (2x + 3)^8 = 0 \text{ or } (x - 1)^2 = 0 \text{ or } (2x + 3)^8 - (x - 1)^5 = 0$$

$$\therefore 2x + 3 = 0 \text{ or } x - 1 = 0 \text{ or } (2x + 3)^8 = (x - 1)^5$$

$$\therefore 2x = -3 \text{ or } x = 1 \text{ or } 2x + 3 = x - 1$$

$$\therefore x = -\frac{3}{2} \text{ or } x = 1 \text{ or } x = -4$$

Substituting the value of x in the given equation verify that the equation is satisfied.

(iv) Solve. $4^x - 3 \cdot 2^x + 2 = 0$

Solution : $4^x - 3 \cdot 2^x + 2 = 0$

$$\therefore (2^2)^x - 3 \cdot 2^x + 2 = 0$$

$$\therefore (2^x)^2 - 3 \cdot 2^x + 2 = 0 \dots \dots \left[\because (a^m)^n = (a^n)^m \right]$$

Substituting $a = 2^x$ in the above equation, we get

$$a^2 - 3a + 2 = 0$$

$$\therefore (a - 2)(a - 1) = 0$$

$$\therefore a = 2 \text{ or } a = 1.$$

Substituting the value of a , we get

$$2^x = 2 \text{ or } 2^x = 1$$

$$\text{i.e. } 2^x = 2^1 \text{ or } 2^x = 2^0 \dots \dots (\because a^0 = 1, a \neq 0).$$

$$\therefore x = 1 \text{ or } x = 0$$

Substituting these values in the given equation, verify that the equation is satisfied.

(v) Solve. $\sqrt[3]{x} - 4 + 3x^{-\frac{1}{3}} = 0$

Solution : $\sqrt[3]{x} - 4 + 3x^{-\frac{1}{3}} = 0 \quad \therefore x^{\frac{1}{3}} - 4 + 3 \times \frac{1}{x^{\frac{1}{3}}} = 0$

Substituting $x^{\frac{1}{3}} = a$ in the equation, we get

$$a - 4 + 3 \frac{1}{a} = 0 \quad \therefore a^2 - 4a + 3 = 0 \quad \therefore (a - 3)(a - 1) = 0$$

$$\therefore a = 3 \text{ or } a = 1$$

Substituting the value of a we get

$$x^{\frac{1}{3}} = 3 \text{ or } x^{\frac{1}{3}} = 1$$

$$\therefore x = 3^3 = 27 \text{ or } x = 1$$

Substituting these values in the given equation, verify that the equation is satisfied.

§ 5.11 Exponential Numbers and Order :

(1) We know that $5 > 2$. If the numbers 5 and 2 are raised to equal powers, will the order relation of the powers remain the same as between the bases ?

i.e. $5 > 2$, in this case will the order relation between $5^3, 2^3$; $5^{-2}, 2^{-2}$; $5^2, 2^2$; $5^{-4}, 2^{-4}$; $5^4, 2^4$ etc. be the same as the order relation between 5 and 2? To know this let us actually calculate and find the order relation.

(a) $5 > 2$ (i) $5^3 = 125$, $2^3 = 8 \quad \therefore 5^3 > 2^3$

(ii) $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$, $2^{-3} = \frac{1}{2^3} = \frac{1}{8} \quad \therefore 5^{-3} < 2^{-3}$

(b) $7 > 5$ (iii) $7^2 = 49$, $5^2 = 25 \quad \therefore 7^2 > 5^2$

(iv) $7^{-4} = \frac{1}{7^4} = \frac{1}{2401}$, $5^{-4} = \frac{1}{5^4} = \frac{1}{625} \quad \therefore 7^{-4} < 5^{-4}$

(c) $5 > 3$ (v) $5^4 = 625$, $3^4 = 81 \quad \therefore 5^4 > 3^4$

(vi) $5^{-4} = \frac{1}{5^4} = \frac{1}{625}$, $3^{-4} = \frac{1}{3^4} = \frac{1}{81} \quad \therefore 5^{-4} < 3^{-4}$

(d) $8 > 5$ (vii) $8^2 = 64$, $5^2 = 25 \quad \therefore 8^2 > 5^2$

(viii) $8^{-3} = \frac{1}{8^3} = \frac{1}{512}$, $5^{-3} = \frac{1}{5^3} = \frac{1}{125} \quad \therefore 8^{-3} < 5^{-3}$

(e) $7 > 2$ (ix) $7^3 = 343$, $2^3 = 8 \quad \therefore 7^3 > 2^3$

(x) $7^{-4} = \frac{1}{7^4} = \frac{1}{2401}$, $2^{-4} = \frac{1}{2^4} = \frac{1}{16} \quad \therefore 7^{-4} < 2^{-4}$

Observe the above examples carefully. When does the order relation between the powers remain the same as that between the bases? When does it reverse?

From the above illustration we arrive at the following general result—

(i) If $a, b > 0$, $x > 0$ and $a > b$ then $a^x > b^x$

(ii) If $a, b > 0$, $x < 0$ and $a > b$ then $a^x < b^x$

(2) Similarly, from the above examples you will find that the converse of the above statement also is true.

i.e. (iii) If $a, b > 0$, $x > 0$ and $a^x > b^x$ then $a > b$.

(iv) If $a, b > 0$, $x < 0$ and $a^x > b^x$ then $a < b$.

Now, let us consider some problems based on the above results.

(i) Which number is greater $2^{\frac{1}{2}}$ or $3^{\frac{1}{3}}$?

Solution: As the numbers stand it is not possible to compare them. Because

neither the base nor the indices of the numbers $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$ are equal. If we can convert these numbers into numbers having equal indices, we can use the above results and decide which of the two numbers is greater.

$$2^{\frac{1}{2}} = 2^{\frac{3}{6}} = (2^3)^{\frac{1}{6}} = 8^{\frac{1}{6}} \text{ and } 3^{\frac{1}{3}} = 3^{\frac{2}{6}} = (3^2)^{\frac{1}{6}} = 9^{\frac{1}{6}}$$

$$8 > 0, 9 > 0 \text{ and } 9 > 8 \text{ is obvious.}$$

Hence using the result 'If $a, b > 0$, $x > 0$ and $a > b$ then $a^x > b^x$, we get $9^{\frac{1}{6}} > 8^{\frac{1}{6}}$ i.e. $3^{\frac{1}{3}} > 2^{\frac{1}{2}}$.

Using this method, it is possible to decide the order of the numbers having exponents.

(ii) Write the following numbers in the ascending order of their magnitudes.

$$2^{-\frac{3}{2}}, 3^{-\frac{2}{3}} \text{ and } 5^{-\frac{1}{3}}.$$

Solution:

$$2^{-\frac{3}{2}} = 2^{-\frac{9}{6}} = (2^9)^{-\frac{1}{6}} = (512)^{-\frac{1}{6}}$$

$$3^{-\frac{2}{3}} = 3^{-\frac{4}{6}} = (3^4)^{-\frac{1}{6}} = (81)^{-\frac{1}{6}}$$

$$5^{-\frac{1}{3}} = 5^{-\frac{2}{6}} = (5^2)^{-\frac{1}{6}} = (25)^{-\frac{1}{6}}$$

Using the result 'If $a, b > 0$, $x < 0$ and $a > b$, then $a^x < b^x$, we get $(512) > (81) > (25)$, hence, $(512)^{-\frac{1}{6}} < (81)^{-\frac{1}{6}} < (25)^{-\frac{1}{6}}$ i.e., $2^{-\frac{3}{2}} < 3^{-\frac{2}{3}} < 5^{-\frac{1}{3}}$.

According to the ascending order the numbers are $2^{-\frac{3}{2}}, 3^{-\frac{2}{3}}, 5^{-\frac{1}{3}}$.

(iii) If $a, b > 0$ and $a^{\frac{5}{2}} > b^{\frac{2}{3}}$, then show that $a^{15} > b^4$.

Solution:

$$a^{\frac{5}{2}} = a^{\frac{15}{6}} = (a^{15})^{\frac{1}{6}} \text{ and } b^{\frac{2}{3}} = b^{\frac{4}{6}} = (b^4)^{\frac{1}{6}}$$

$$\therefore a^{\frac{5}{2}} > b^{\frac{2}{3}} \therefore (a^{15})^{\frac{1}{6}} > (b^4)^{\frac{1}{6}}$$

Using the result 'If $a, b > 0$, $x > 0$ and $a^x > b^x$ then $a > b$ ', we get $a^{15} > b^4$.

(iv) If $x, y > 0$ and $x^{-\frac{2}{3}} < y^{-\frac{1}{6}}$ then show that $x^{10} - y^3 > 0$.

Solution:

$$x^{-\frac{2}{3}} = x^{-\frac{10}{15}} = (x^{10})^{-\frac{1}{15}} \text{ and } y^{-\frac{1}{6}} = y^{-\frac{3}{18}} = (y^3)^{-\frac{1}{18}}$$

$$\therefore x^{-\frac{2}{3}} < y^{-\frac{1}{6}} \therefore (x^{10})^{-\frac{1}{15}} < (y^3)^{-\frac{1}{18}}$$

Now using the result 'If $a, b > 0$, $x < 0$ and $a^x < b^x$ then $a > b$ ', we get
 $x^{10} > y^3 \therefore x^{10} - y^3 > 0$.

(3) If m and n are any real numbers and $m > n$ then what can you say about the order of the numbers x^m and x^n .

To know this study the following examples carefully.

(i) $3^5 = 343$ and $3^3 = 27 \therefore 3^5 > 3^3$.

(ii) $(.2)^3 = 0.08$ and $(.2)^5 = 0.0032 \therefore (.2)^5 < (.2)^3$.

(iii) $2^4 = 16$ and $2^2 = 4 \therefore 2^4 > 2^2$.

(iv) $(.3)^4 = 0.0081$ and $(.3)^2 = .09 \therefore (.3)^4 < (.3)^2$.

(v) $7^4 = 2401$ and $7^2 = 49 \therefore 7^4 > 7^2$.

(vi) $(.5)^4 = .0625$ and $(.5)^2 = .25 \therefore (.5)^4 < (.5)^2$.

Observe the above examples and answer the following questions.

- (i) Is the order of the exponents and the order of the power always same?
- (ii) When the base is greater than 1, is the order of the exponents the same as that of the powers?
- (iii) When the base is less than unity, is the order of the exponents the same as that of the powers?

From the above examples we get the general rule as follows.

(i) If $m > n$ then $a^m > a^n$.

(ii) If $0 < a < 1$ and $m > n$ then $a^m < a^n$.

From the above examples you will easily notice that the converse of the statements (i) and (ii) above is also true.

i.e. (i) If $a^m > a^n$ then $m > n$.

(ii) If $0 < a < 1$ and $a^m > a^n$ then $m < n$.

Now let us solve some examples based on these results.

(i) If $523 = 10^k$ then show that $2 < k < 3$.

Solution:

$$\therefore 100 < 523 < 1000. \therefore 10^2 < 10^k < 10^3.$$

Now using the result 'If $a > 1$ and $a^m > a^n$ then $m > n$ ', we have $2 < k < 3$.

Similarly, show that with respect to base 10, the index of every number between 100 and 1000 lies between 2 and 3.

(ii) If $0.0847 = 10^m$, then show that $-2 < m < -1$.

Solution:

$$\because 0.1 > 0.0847 > 0.01. \therefore 10^{-1} > 10^m > 10^{-2}$$

Now using the result 'If $a > 1$ and $a^m > a^n$ then $m > n$ ', we get
 $-1 > m > -2$. i.e. $-2 < m < -1$.

Similarly, show that with respect to the base 10, the exponent of every number between 0.01 and 0.1 lies between -2 and -1 .

EXERCISE 5.8

1. Write the following numbers in the ascending order.

(i) $2^{\frac{2}{3}}, 5^{\frac{1}{2}}, 7^{\frac{1}{3}}$ (ii) $2^{\frac{1}{3}}, 3^{\frac{1}{4}}, 5^{\frac{1}{5}}$ (iii) $5^{-\frac{1}{3}}, 6^{-\frac{1}{2}}, 7^{-\frac{2}{3}}$ (iv) $2^{-\frac{1}{2}}, 3^{-\frac{1}{3}}, 4^{-\frac{1}{3}}$

2. Solve the following equations.

(i) $3^{2x+1} = 3^{x-1}$ (ii) $5^{y+3} = (25)^{3y-4}$ (iii) $(125)^{4x+3} = (25)^{x-4}$

(iv) $(\sqrt{3})^{x+5} = (\sqrt[3]{3})^{2x+5}$ (v) $2^{x+1} = 4^{3y-4}; 9^{x-y} = (27)^{x+y-1}$

(vi) $a^{x-y-8} = (a^{2x+y-1})^2; (25)^{3x-y+1} = (125)^{x-2y-1}$

(vii) $(3x+7)^2 (4x-7)^7 = (4x-7)^3 (3x+7)^6$

(viii) $(x+1)^6 (2x-1)^{18} = (x+1)^9 (2x-1)^{10}$

(ix) $5^{2x} - 6 \times 5^x + 5 = 0$

(x) $\sqrt{x} - 4 + \frac{3}{\sqrt{x}} = 0$

(xi) $2\sqrt[3]{x} - 7 + 3x^{-\frac{1}{3}} = 0$

(xii) $\left(\frac{a}{b}\right)^{x+1} = \left(\frac{b}{a}\right)^{2x-3}$

3. If $x, y > 1$ and $x^{\frac{2}{3}} > y^{\frac{5}{4}}$ then show that $x^8 > y^{15}$.

4. If $x, y > 1$ and $x^{-\frac{4}{5}} > y^{-\frac{2}{3}}$, then show that $y^{10} - x^{12} > 0$.

5. If $783 = 10^m$, then show that $2 < m < 3$.

6. If $0.283 = 10^n$, then show that $-1 < n < 0$.

EXERCISE 5.9

1. Simplify.

(i) $\frac{(27)^{\frac{2}{3}n} 8^{-\frac{n}{6}}}{(18)^{-\frac{n}{2}}}$

(ii) $\frac{2^{\frac{1}{3}} \times 3^{\frac{1}{6}} \times 5^{\frac{5}{6}} \times 6^{\frac{1}{2}}}{6^{\frac{1}{3}} \times (10)^{\frac{5}{3}} \times (15)^{\frac{1}{3}}}$

(iii) $1 - \left\{ 1 - (1 - x^2)^{-1} \right\}^{-1}$

(iv) $2 \div \left(2 \div 2^{\frac{1}{3}} \right)^{\frac{1}{2}}$

(v) $\sqrt[n]{\frac{32}{2^{5+n}}}$

(vi) $8^{\frac{4}{3}} + \sqrt[3]{2 \times 4^{-5}} - \frac{\sqrt[3]{8}}{4^{-\frac{2}{3}}} - (32)^{-\frac{3}{5}}$

$$(vii) \frac{x^{\frac{1}{2}} y^{\frac{1}{3}}}{z^{\frac{1}{6}}} \div \left\{ \frac{(xy)^{\frac{1}{2}}}{z^{\frac{1}{2}}} \times \left(\frac{y}{x}\right)^{\frac{2}{3}} \times z^{-\frac{2}{3}} \right\}$$

$$(viii) \left[1 - \left\{ 1 - (1-x^2)^{-1} \right\}^{-1} \right]^{-\frac{1}{2}}$$

$$(x) \frac{(3)(3^n)}{(3^n)^{n-1}} \div \frac{9^{n+1}}{(3^{n-1})^{n+1}}$$

2. Solve the equations.

$$(i) 2^{x+7} = 4^{x+2}$$

$$(ii) (\sqrt[3]{25})^{2x+1} = (\sqrt[5]{125})^{x+1}$$

$$(iii) 2^{3x-5} a^{x-2} = 2^{x-3} \times 2a^{1-x} \text{ if } (a \neq 0)$$

$$(iv) (x-1)^4 (2x+1)^7 = (x-1)^9 (2x+1)^2$$

$$(v) x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$$

$$(vi) x^{-1} - 3x^{-\frac{1}{2}} + 2 = 0$$

3. Expand.

$$(i) \left(x^{\frac{2}{3}} + x^{\frac{1}{3}}\right)^2$$

$$(ii) \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)^2$$

$$(iii) \left(x^{\frac{2}{3}} - x^{-\frac{2}{3}}\right)^3$$

$$(iv) \left(x^{\frac{1}{2}} y^{-\frac{1}{2}} + x^{-\frac{1}{2}} y^{\frac{1}{2}}\right)^2$$

$$(v) \left(3^{\frac{1}{3}} + 3^{-\frac{1}{3}}\right)^3$$

4. Express the following expressions in the form of perfect square or cube as the case may be.

$$(i) x^{\frac{1}{2}} + 2 + x^{-\frac{1}{2}}$$

$$(ii) x^2 + 3\left(x^{\frac{2}{3}} + x^{-\frac{2}{3}}\right) + x^{-2}$$

$$(iii) ab^{-1} - 2 + a^{-1}b$$

$$(iv) xy^{-1} - 3\left(x^{\frac{1}{3}}y^{-\frac{1}{3}} - x^{-\frac{1}{3}}y^{\frac{1}{3}}\right) - x^{-1}y$$

5. Express in the form $a^2 - b^2$ and write 3 factors.

$$(i) x^{-1} - y^{-1}$$

$$(ii) x^{\frac{1}{2}} - y^{\frac{1}{2}}$$

$$(iii) x^{\frac{2}{3}} - y^{\frac{2}{3}}$$

6. Express in the form $a^3 - b^3$ or $a^3 + b^3$ and factorise.

$$(i) x^{-1} - y^{-\frac{1}{2}}$$

$$(ii) x^{\frac{1}{2}} - y^{-\frac{1}{2}}$$

$$(iii) x^{\frac{1}{3}} + y^{-1}$$

$$(iv) x^{\frac{2}{3}} + y^{\frac{2}{3}}$$

7. Write the following numbers in descending order of magnitude.

$$(i) \sqrt[4]{6}, \sqrt{2}, \sqrt[4]{4} \quad (ii) \sqrt[4]{3}, \sqrt[4]{10}, \sqrt[4]{125} \quad (iii) 5^{\frac{1}{2}}, 4^{\frac{1}{3}}, (10)^{\frac{1}{4}}$$

Chapter 6 : LOGARITHMS

You have learnt how to multiply and divide numbers involving indices. The concept of logarithm is closely connected with the concept of indices. In this Chapter you will study logarithms.

§ 6.1

Use of logarithms helps us to simplify complex numerical calculations. Because of this it serves as a very useful tool in different branches of mathematics and science.

Study the following examples to know logarithms.

Consider $3^4 = 81$. We call 3 as the base and 4 the index. Here we have expressed the number 81 as the power of 3, the index being 4. As an alternative way, we say that 4 is the logarithm of 81 to the base 3. Note that here we identify 'index' with 'logarithm' while expressing 81 as the power of the base 3, the index being 4. So we shall say that the logarithm of 81 to 3 as the base is 4.

In $10^3 = 1000$, we say that logarithm of 1000 to 10 as base is 3. We write this as ' $\log_{10} 1000 = 3$ ' and read as 'log 1000 to the base 10 is equal to 3'.

In $2^5 = 32$, we say that logarithm of 32 to 2 as the base, is 5. We write this as ' $\log_2 32 = 5$ ' and read as 'log 32 to the base 2 is equal to 5.'

Similarly

- | | |
|-----------------------|---------------------------------|
| (i) $10^2 = 100$ | $\therefore 2 = \log_{10} 100$ |
| (ii) $3^5 = 243$ | $\therefore 5 = \log_3 243$ |
| (iii) $10^{-1} = 0.1$ | $\therefore -1 = \log_{10} 0.1$ |
| (iv) $a^3 = 8$ | $\therefore 3 = \log_a 8$ |
| (v) $10^x = y$ | $\therefore x = \log_{10} y$ |

Observe that the 'index' and 'logarithm' are in a way the same concepts. The difference lies in the form of expression. To say that $4^3 = 64$ is the same as saying $3 = \log_4 64$.

To find $\log_5 625$, is to find the exponent of the base 5 which will give rise to the number 625.

From this we define logarithm of a given number to a given base as follows.

If $a^x = y$ then $x = \log_a y$, $a > 0$ and $a \neq 1$.

In defining 'log' we have to impose some restrictions, such as $a > 0$, $a \neq 1$. Note the reasons of these conditions which are discussed below.

(i) Recall that in $a^x = y$, the base 'a' is positive. Hence the restriction that the base of the logarithm should be positive.

(ii) For any real number x , $1^x = 1$. Hence no number other than 1 itself can be expressed as power of 1. Therefore 1 cannot be taken as a base for logarithm.

Henceforth it will be understood that the base of logarithm fulfills these two conditions.

Using the definition of logarithm we have expressed exponents of numbers in terms of logarithms. From the same definition, we can express a number in the logarithmic form in terms of exponents of its base.

$$\begin{array}{ll} \text{(i)} \log_2 8 = 3 & \therefore 2^3 = 8 \\ \text{(ii)} \log_{10} 100 = 2 & \therefore 10^2 = 100 \\ \text{(iii)} \log_{10} 0.01 = -2 & \therefore 10^{-2} = 0.01 \\ \text{(iv)} \log_{10} m = x & \therefore (10)^x = m \\ \text{(v)} \log_a 5 = x & \therefore a^x = 5 \end{array}$$

EXERCISE 6.1

- Express the following numbers in the logarithmic form.
 (i) $10^2 = 100$ (ii) $5^3 = 125$ (iii) $2^x = 16$ (iv) $a^3 = 27$ (v) $10^y = m$
 (vi) $5^0 = 1$ (vii) $7^1 = 7$ (viii) $a = b^x$ (ix) $m = n^{x+y}$ (x) $10^{1.5776} = 37.81$
 (xi) $0.001 = 10^{-3}$ (xii) $10^{1.3010} = 20$
- Express the following numbers in the exponential form.
 (i) $\log_5 625 = 4$ (ii) $\log_a 5 = k$ (iii) $\log_5 5 = 1$
 (iv) $\log_7 1 = 0$ (v) $\log_{10} 0.01 = -2$ (vi) $\log_k (x+y) = c$
 (vii) $\log_6 m = p$ (viii) $\log_{10} 0.1 = -1$ (ix) $\log_{10} 1000 = 3$
 (x) $\log_{10} 2.378 = 0.3762$ (xi) $\log_{10} 37.81 = 1.5776$
- Find the values of—
 e.g. $\log_7 343$. Let $\log_7 343 = x$. $\therefore$ By definition $7^x = 343$. $\therefore x = 3$.
 (i) $\log_{10} 100$ (ii) $\log_3 81$ (iii) $\log_7 49$ (iv) $\log_{25} 625$ (v) $\log_2 64$
 (vi) $\log_4 64$ (vii) $\log_8 64$ (viii) $\log_{64} 64$ (ix) $\log_{10} 0.1$ (x) $\log_{10} 0.01$
 (xi) $\log_{\sqrt{2}} 2$ (xii) $\log_{\sqrt{3}} 9$ (xiii) $\log_{10} 1000$ (xiv) $\log_{17} 1$ (xv) $\log_{23} 23$
 (xvi) $\log_{5\sqrt{3}} 125$
- Simplify.
 (i) $\log_2 8 + \log_5 25 - \log_3 9$ (ii) $5 \log_7 49 - 3 \log_2 8$
 (iii) $3 \log_5 5 - 7 \log_2 16 - 2 \log_6 36$ (iv) $\log_2 (1 + 2 + 5) + \log_7 7$
 (v) $\log_{10} 100 \times \log_3 27$ (vi) $732 \log_5 27 \times \log_7 1$
 (vii) $\log_{10} 100 + \log_{10} 10 + \log_{10} 1 + \log_{10} 0.1 + \log_{10} 0.01$
 (viii) $(2 \log_{11} 121) \times (3 \log_{12} 144)$ (ix) $\frac{8 \log_7 7}{4 \log_3 3}$ (x) $\frac{4 \log_5 25}{3 \log_3 81}$
- Find the values of l. h. s. and r. h. s. of each example and verify whether the two sides are equal.
 (i) $\log_2 (4 \times 16) = \log_2 4 + \log_2 16$
 l. h. s. $= \log_2 (4 \times 16) = \log_2 64 = 6$.. (1)
 r. h. s. $= \log_2 4 + \log_2 16 = 2 + 4 = 6$.. (2)
 From (1) and (2), we have $\log_2 (4 \times 16) = \log_2 4 + \log_2 16$

$$(ii) \log_2 \left(\frac{32}{4} \right) = \frac{\log_2 32}{\log_2 4}$$

$$\text{l. h. s.} = \log_2 \left(\frac{32}{4} \right) = \log_2 8 = 3 \dots (1)$$

$$\text{r. h. s.} = \frac{\log_2 32}{\log_2 4} = \frac{5}{2} \dots (2)$$

From (1) and (2), we have $\log_2 \left(\frac{32}{4} \right) \neq \frac{\log_2 32}{\log_2 4}$

$$(i) \log_3(3 \times 9) = \log_3 3 + \log_3 9 \quad (ii) \log_3(3 \times 9) = (\log_3 3) \times (\log_3 9)$$

$$(iii) \log_2(4 \times 32) = (\log_2 4) \times (\log_2 32) \quad (iv) \log_2(4 \times 32) = \log_2 4 + \log_2 32$$

$$(v) \log_5 \left(\frac{125}{25} \right) = \log_5 125 - \log_5 25 \quad (vi) \log_5 \left(\frac{125}{25} \right) = \frac{\log_5 125}{\log_5 25}$$

$$(vii) \log_2 \left(\frac{128}{16} \right) = \frac{\log_2 128}{\log_2 16} \quad (viii) \log_2 \left(\frac{128}{16} \right) = \log_2 128 - \log_2 16$$

$$(ix) \log_2 8^2 = 2 \log_2 8 \quad (x) \log_3 9^2 = 2 \log_3 9 \quad (xi) \log_5 (25)^2 = 5 \log_5 25$$

$$(xii) \log_3 \left(\frac{243}{9} \right) = \frac{\log_3 243}{\log_3 9} \quad (xiii) \log_3 \left(\frac{243}{9} \right) = \log_3 243 - \log_3 9$$

$$(xiv) \log_5 (25 \times 5) = \log_5 25 + \log_5 5 \quad (xv) \log_5 (25 \times 5) = (\log_5 25) \times (\log_5 5)$$

$$(xvi) \log_5 (1 + 2 + 3) = \log_5 (1 \times 2 \times 3) \quad (xvii) \log_2 (4 + 8) = \log_2 4 + \log_2 8$$

$$(xviii) \log_2 (4 \times 8) = \log_2 4 + \log_2 8 \quad (xix) \log_3 (27 + 9) = \log_3 27 + \log_3 9$$

$$(xx) \log_3 (27 \times 9) = \log_3 27 + \log_3 9 \quad (xxi) \log_2 (8 - 4) = \log_2 8 - \log_2 4$$

$$(xxii) \log_2 \left(\frac{8}{4} \right) = \log_2 8 - \log_2 4 \quad (xxiii) \log_3 \left(\frac{27}{9} \right) = \log_3 27 - \log_3 9$$

$$(xxiv) \log_3 (27 - 9) = \log_3 27 - \log_3 9$$

(xxv) Numbers in the form of logarithm are given in the columns below. State which numbers from the right column are equal to the numbers from the left column.

For this, use the above examples from (i) to (xxiv) in which l. h. s. is equal to r. h. s.

$$(1) \log_a m^n$$

$$(2) \log_a (m + n)$$

$$(3) \log_a mn$$

$$(4) \log_a \left(\frac{m}{n} \right)$$

$$(5) \log_a (m - n)$$

$$(A) \log_a m + \log_a n$$

$$(B) (\log_a m) \times (\log_a n)$$

$$(C) n \log_a m$$

$$(D) \frac{\log_a m}{\log_a n}$$

$$(E) \log_a m - \log_a n$$

(F) None of the above A, B, C, D, E alternatives.

§ 6.2 $\log_a a = 1$ and $\log_a 1 = 0$; $a > 0$, $a \neq 1$:

(1) Let us now consider as to what would be the value of the logarithm of a number, if the base of the logarithm is the given number itself. For this let us find the value of $\log_7 7$.

Suppose $\log_7 7 = x$, then $7^x = 7$. But $7^0 = 7^1$. $\therefore 7^x = 7^1$. $\therefore x = 1$.

Similarly it can be verified that the values of $\log_5 5$, $\log_4 4$, $\log_{723} 723$, etc. also come out to be 1.

From these examples, we can say that in general $\log_a a = 1$. i.e. Logarithm of any number to itself as the base is equal to unity.

Let us now establish this general result.

Suppose $\log_a a = k$.

If $a^x = y$ then $x = \log_a y$. Using this definition we have $a^k = a$.

i.e. $a^k = a^1$. $\therefore k = 1$. $\therefore \log_a a = 1$. Provided $a \neq 1$.

(2) Let us now consider the value of the logarithm of unity to any base. For this, consider the value of $\log_8 1$.

Suppose $\log_8 1 = x$, then $8^x = 1$. The value of 8^x is one if and only if $x = 0$.

$\therefore x = 0$ $\therefore \log_8 1 = 0$.

Similarly it can be shown that $\log_5 1$, $\log_7 1$, $\log_{13} 1$ etc. come out to be zero.

From these examples, we can say that $\log_a 1 = 0$, ($a \neq 1$, $a > 0$).

i.e. logarithm of unity to any base is equal to zero.

Now we shall prove this general result.

Suppose $\log_a 1 = k$; $a > 0$, $a \neq 1$.

$\therefore$ By the definition of logarithm, we have $a^k = 1$.

i.e. $a^k = a^0$. ($\because a^0 = 1$) $\therefore k = 0$ $\therefore \log_a 1 = 0$.

§ 6.3 If $\log_a x = \log_a y$, then $x = y$:

(1) If the logarithm of the two given numbers to the same base are equal then are these numbers also equal? i.e. If $\log_a x = \log_a y$ then what is the relation between x and y ?

Suppose $\log_a x = \log_a y = k$

$$\therefore \log_a x = k \quad \therefore x = a^k \quad \dots \dots \dots (1)$$

$$\therefore \log_a y = k \quad \therefore y = a^k \quad \dots \dots \dots (2)$$

From (1) and (2) we have $x = y$.

$\therefore$ If $\log_a x = \log_a y$ then $x = y$.

(2) If $\log_a x = \log_a y$ then $x = y$. Is the converse of this statement true? Write the converse of this statement and its proof also.

Study the following examples based on these properties.

(1) Solve. $\log_a x^2 = \log_a (3x - 2)$.

Solution : $\log_a x^2 = \log_a (3x - 2)$.

'If $\log_a x = \log_a y$ then $x = y$ '. Using this result, we have $x^2 = 3x - 2$.

$$\therefore x^2 - 3x + 2 = 0 \quad \therefore (x - 2)(x - 1) = 0$$

$$\therefore (x - 2) = 0 \text{ or } (x - 1) = 0 \quad \therefore x = 2 \text{ or } x = 1$$

Substituting these values of x in the given equation verify that the equation is satisfied.

(2) Solve. $\log_9 (x^2 - 8x) = 1$.

Solution : $\log_9 (x^2 - 8x) = 1 \quad \therefore x^2 - 8x = 9^1$
 $\therefore x^2 - 8x - 9 = 0 \quad \therefore (x - 9)(x + 1) = 0$
 $\therefore x - 9 = 0 \text{ or } x + 1 = 0 \quad \therefore x = 9 \text{ or } x = -1$

Substituting these values of x in the given equation verify that the equation is satisfied.

EXERCISE 6.2

- Find the value of x in the following examples.
 - $\log_a x^2 = \log_a (5x - 6)$
 - $\log_a (x^2 + 2x - 7) = 0$
 - $\log_x^2 (7x - 10) = 1$
 - $\log_2 x = 3$
 - $\log_2 \{ \log_2 x \} = 1$
 - $\log_a (7x + 13) = \log_a (2x - 5)$
- Eliminating logarithmic form, write the relation between a and b .
 - $\log_m (2a - 3b) = \log_m (a + b)$
 - $\log_m (2a^2 - ab + 3b^2) = \log_m (a^2 - ab + 2b^2)$

§ 6.4

Now we shall study the laws of logarithms which are useful in simplifying calculations.

(I) $\log_a (m \times n) = \log_a m + \log_a n$.

$\log_2 (5 \times 3) = \log_2 5 + \log_2 3$.

Let us test whether this statement is true.

Suppose $\log_2 5 = x$ and $\log_2 3 = y$ (1)

Then $2^x = 5$ and $2^y = 3$.

$\therefore 5 \times 3 = 2^x \times 2^y = 2^{x+y}$

$\therefore 2^{x+y} = (5 \times 3)$.

Converting this into logarithmic form we have

$x + y = \log_2 (5 \times 3)$ (2)

From (1) substituting the values of x and y in (2), we get

$\log_2 5 + \log_2 3 = \log_2 (5 \times 3)$

$\therefore \log_2 (5 \times 3) = \log_2 5 + \log_2 3$.

Similarly taking different numbers you can verify that the logarithm of the product of two numbers is equal to the sum of the logarithms, base being the same.

From this we get the following general result.

$\log_a (m \times n) = \log_a m + \log_a n$

Proof: Suppose $\log_a m = x$ and $\log_a n = y$ (A)

$\therefore a^x = m$ and $a^y = n \quad \therefore m \times n = a^x \times a^y = a^{x+y}$.

$\therefore$ From the definition of logarithm, we have

$x + y = \log_a (m \times n)$ (B)

Substituting in (B) the values of x and y from (A), we get

$\log_a m + \log_a n = \log_a (m \times n)$

$\therefore \log_a (m \times n) = \log_a m + \log_a n$.

Study the following examples based on this result.

- (1) Simplify : (i) $\log_{10}(25) + \log_{10}4$ (ii) $\log_2 5 + \log_2 7 + \log_2 3 + \log_2 9$.

Solution :

$$(i) \log_{10}(25) + \log_{10}4 = \log_{10}(25 \times 4) = \log_{10}100 = 2$$

$$(ii) \log_2 5 + \log_2 7 + \log_2 3 + \log_2 9.$$

$$= (\log_2 5 + \log_2 9) + (\log_2 3 + \log_2 7) \quad \dots (\text{Associative property}).$$

$$= \log_2(5 \times 9) + \log_2(3 \times 7)$$

$$= \log_2(5 \times 9 \times 3 \times 7) = \log_2(945).$$

From the given example and the 2nd step from bottom in this example, we have $\log_2 5 + \log_2 7 + \log_2 3 + \log_2 9 = \log_2(5 \times 7 \times 3 \times 9)$.

This means that the law— $\log_a(m \times n) = \log_a m + \log_a n$ can be applied even when there is a product of more than two numbers.

i.e. $\log_a(p \times q \times r \times s \times \dots) = \log_a p + \log_a q + \log_a r + \log_a s + \dots$

Let us study the proof of this property.

$$\log_a p + \log_a q + \log_a r + \log_a s + \dots$$

$$= (\log_a p + \log_a q) + (\log_a r + \log_a s) + \dots (\text{Associative property}).$$

$$= \log_a(p \times q) + \log_a(r \times s) + \dots$$

$$= \log_a(p \times q \times r \times s) + \dots$$

$$\therefore \log_a(p \times q \times r \times s \times \dots) = \log_a p + \log_a q + \log_a r + \log_a s + \dots$$

- (2) Express in form of the sum of logarithms.

$$\log_{10}(37.8 \times 24.3 \times 7^3).$$

$$\text{Solution : } \log_{10}(37.8 \times 24.3 \times 7^3)$$

$$= \log_{10}(37.8) + \log_{10}(24.3) + \log_{10}(7^3)$$

$$= \log_{10}(37.8) + \log_{10}(24.3) + \log_{10}(7 \times 7 \times 7)$$

$$= \log_{10}(37.8) + \log_{10}(24.3) + \log_{10}7 + \log_{10}7 + \log_{10}7$$

$$= \log_{10}(37.8) + \log_{10}(24.3) + 3 \log_{10}7.$$

- (3) Find the value of (3.712×5.017) if,

$$\log_{10}(3.712) = 0.5696, \log_{10}(5.017) = 0.7004, \log_{10}(18.62) = 1.2700.$$

Solution :

$$\log_{10}(3.712 \times 5.017) = \log_{10}(3.712) + \log_{10}(5.017)$$

$$= 0.5696 + 0.7004 = 1.2700 = \log_{10}(18.62)$$

$$\therefore \log_{10}(3.712 \times 5.017) = \log_{10}(18.62) \therefore 3.712 \times 5.017 = 18.62.$$

EXERCISE 6.3

1. Simplify.

$$(i) \log_6 9 + \log_6 4 \quad (ii) \log_5 25 + \log_5 7 \quad (iii) \log_5\left(\frac{8}{3}\right) + \log_5\left(\frac{3}{5}\right)$$

$$(iv) \log_a 7 + \log_a 4 + \log_a 8$$

$$(v) \log_m\left(\frac{2}{5}\right) + \log_m\left(\frac{5}{7}\right) + \log_m\left(\frac{7}{13}\right) + \log_m\left(\frac{13}{2}\right)$$

$$(vi) \log_n x + \log_n y + \log_n z \quad (vii) \log_x a + \log_x b$$

$$(viii) \log_a\left(\frac{x}{y}\right) + \log_a\left(\frac{y}{z}\right) + \log_a\left(\frac{z}{x}\right)$$

2. Find the prime factors and express the logarithm of the given number in terms of sum of the logarithms of two or more numbers.

- (i) $\log_2 15$ (ii) $\log_4 21$ (iii) $\log_5 66$ (iv) $\log_5 7^2$ (v) $\log_7 5^3$
 (vi) $\log_2 32$ (vii) $\log_5 (a \times b)$ (viii) $\log_x [(a+b)(c+d)]$ (ix) $\log_m (a^2 - b^2)$
 (x) $\log_m \left(\frac{x}{y}\right)$ (xi) $\log_x \left(\frac{a+b}{c+d}\right)$ (xii) $\log_n (3.78 \times 5.13)$
 (xiii) $\log_a (7.132 \times 4.7 \times 81.23)$ (xiv) $\log_{10} (3.4)^3$

3. $\log_{10} (31.78) = 1.5022$; $\log_{10} (2.34) = 0.3692$;
 $\log_{10} (74.37) = 1.8714$; $\log_{10} (3.47) = 0.5403$;
 $\log_{10} (2.313) = 0.3642$; $\log_{10} (8.026) = 0.9045$;
 $\log_{10} (5.413) = 0.7334$; $\log_{10} (4.782) = 0.6796$;
 $\log_{10} (2.001) = 0.3012$; $\log_{10} (51.78) = 1.7142$;
 $\log_{10} (4.314) = 0.6349$; $\log_{10} (18.61) = 1.2698$;
 $\log_{10} (2.781) = 0.4442$; $\log_{10} (166.3) = 2.2210$;

Using these, find the values of—

- (i) 31.78×2.34 (ii) 3.47×2.313 (iii) $5.413 \times 4.782 \times 2.001$
 (iv) $(4.314)^2$ (v) $(2.781)^5$.

§ 6.5 (II) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$:

$\log_2 \left(\frac{7}{5}\right) = \log_2 7 - \log_2 5$. Let us see whether this statement is true.

Suppose $\log_2 7 = x$ and $\log_2 5 = y$ (1)

$$\therefore 2^x = 7 \text{ and } 2^y = 5$$

$$\therefore \frac{7}{5} = \frac{2^x}{2^y} = 2^{x-y}$$

$$\therefore 2^{x-y} = \left(\frac{7}{5}\right)$$

From the definition of logarithm, we have

$$x - y = \log_2 \left(\frac{7}{5}\right) \text{(2)}$$

Substituting in (2) the values of x and y from (1), we get

$$\log_2 7 - \log_2 5 = \log_2 \left(\frac{7}{5}\right)$$

$$\therefore \log_2 \left(\frac{7}{5}\right) = \log_2 7 - \log_2 5$$

Similarly by taking different numbers you can verify that the above property holds good.

From these we state the general rule : $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$.

Now, let us study the proof of this property.

Suppose $\log_a m = x$ and $\log_a n = y$

....(A)

$$\therefore a^x = m \text{ and } a^y = n \quad \therefore \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$$

$$\therefore a^{x-y} = \left(\frac{m}{n}\right)$$

Using the definition of logarithm, we have $x - y = \log_a \left(\frac{m}{n}\right)$

....(B)

Substituting in (B) the values of x and y from (A), we get

$$\log_a m - \log_a n = \log_a \left(\frac{m}{n}\right)$$

$$\therefore \log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Study the following examples based on this property.

(1) Simplify. $\log_3 18 + \log_3 6 - \log_3 12$.

Solution :

$$\log_3 18 + \log_3 6 - \log_3 12$$

$$= (\log_3 18 + \log_3 6) - \log_3 12$$

.... (Associative property)

$$= \log_3 (18 \times 6) - \log_3 12 = \log_3 \left(\frac{18 \times 6}{12}\right)$$

$$= \log_3 9 = 2.$$

(2) Express in the form of the sum or the difference of the logarithms.

$$\log_{10} \left(\frac{3 \cdot 7 \times 5 \cdot 7}{2 \cdot 3 \times 2 \cdot 2}\right)$$

$$\text{Solution : } \log_{10} \left(\frac{3 \cdot 7 \times 5 \cdot 7}{2 \cdot 3 \times 2 \cdot 2}\right)$$

$$= \log_{10} (3 \cdot 7 \times 5 \cdot 7) - \log_{10} (2 \cdot 3 \times 2 \cdot 2).$$

$$= \log_{10} (3 \cdot 7) + \log_{10} (5 \cdot 7) - [\log_{10} (2 \cdot 3) + \log_{10} (2 \cdot 2)]$$

$$= \log_{10} (3 \cdot 7) + \log_{10} (5 \cdot 7) - \log_{10} (2 \cdot 3) - \log_{10} (2 \cdot 2).$$

(3) Given that $\log_{10} (4 \cdot 71) = 0 \cdot 6721$; $\log_{10} (2 \cdot 3) = 0 \cdot 3617$;

$$\log_{10} (5 \cdot 73) = 0 \cdot 7582 \text{ and } \log_{10} (1 \cdot 887) = 0 \cdot 2756$$

$$\text{find the value of } \frac{4 \cdot 71 \times 2 \cdot 3}{5 \cdot 73}$$

$$\text{Solution : } \log_{10} \left(\frac{4 \cdot 71 \times 2 \cdot 3}{5 \cdot 73}\right)$$

$$= \log_{10} (4 \cdot 71 \times 2 \cdot 3) - \log_{10} (5 \cdot 73)$$

$$= \log_{10} (4 \cdot 71) + \log_{10} (2 \cdot 3) - \log_{10} (5 \cdot 73)$$

$$= 0 \cdot 6721 + 0 \cdot 3617 - 0 \cdot 7582 = 0 \cdot 2756 = \log_{10} (1 \cdot 887)$$

$$\therefore \log_{10} \left(\frac{4 \cdot 71 \times 2 \cdot 3}{5 \cdot 73}\right) = \log_{10} (1 \cdot 887)$$

$$\therefore \frac{4 \cdot 71 \times 2 \cdot 3}{5 \cdot 73} = 1 \cdot 887$$

EXERCISE 6.4

1. Simplify.

$$\begin{aligned} & \text{(i) } \log_2 8 - \log_2 4 \quad \text{(ii) } \log_5 17 - \log_5 7 \quad \text{(iii) } \log_m a - \log_m b \\ & \text{(iv) } \log_a x - \log_a y - \log_a z \quad \text{(v) } \log_m (a^2 - b^2) - \log_m (a + b) \quad \text{(vi) } \log_2 \left(\frac{1}{32} \right) \\ & \text{(vii) } \log_a \left(\frac{x}{y} \right) - \log_a x + \log_a \left(\frac{1}{y} \right) \quad \text{(viii) } \log_k (a^3 + b^3) - \log_k (a^3 - ab + b^2) + \\ & \log_k (a - b) \quad \text{(ix) } \log_k \left(\frac{a}{b} \right) - \log_k \left(\frac{b}{a} \right) \quad \text{(x) } \log_k x - \log_k \left(\frac{1}{x} \right). \end{aligned}$$

2. Express as the sum or difference of the logarithms.

$$\begin{aligned} & \text{(i) } \log_a \left(\frac{13}{7} \right) \quad \text{(ii) } \log_a \left(\frac{x}{y} \right) \quad \text{(iii) } \log_{10} \left(\frac{27 \cdot 32}{8 \cdot 437} \right) \quad \text{(iv) } \log_{10} \left(\frac{2 \times 7}{5} \right) \\ & \text{(v) } \log_a \left(\frac{x \times y}{z} \right) \quad \text{(vi) } \log_{10} \left(\frac{2 \cdot 314 \times 37 \cdot 99}{48 \cdot 33} \right) \quad \text{(vii) } \log_{10} \left(\frac{3}{8 \times 5} \right) \\ & \text{(viii) } \log_k \left(\frac{a}{b \times c} \right) \quad \text{(ix) } \log_{10} \left(\frac{99 \cdot 07}{0 \cdot 03 \times 0 \cdot 532} \right) \quad \text{(x) } \log_{10} \left(\frac{3 \times 5}{7 \times 8} \right) \\ & \text{(xi) } \log_{10} \left(\frac{8 \cdot 33 \times 7 \cdot 5}{2 \cdot 8 \times 0 \cdot 38} \right) \quad \text{(xii) } \log_{10} \left(\frac{0 \cdot 03 \times 7 \cdot 089}{2 \cdot 7 \times 0 \cdot 532} \right) \end{aligned}$$

$$\begin{aligned} 3. \quad & \log_{10} (27 \cdot 82) = 1 \cdot 4443; & \log_{10} (3 \cdot 537) = 0 \cdot 5487; \\ & \log_{10} (7 \cdot 863) = 0 \cdot 8956; & \log_{10} (4 \cdot 27) = 0 \cdot 6304; \\ & \log_{10} (8 \cdot 271) = 0 \cdot 9176; & \log_{10} (5 \cdot 235) = 0 \cdot 7189; \\ & \log_{10} (6 \cdot 747) = 0 \cdot 8291; & \log_{10} (84 \cdot 82) = 1 \cdot 9285; \\ & \log_{10} (52 \cdot 63) = 1 \cdot 7272; & \log_{10} (1 \cdot 583) = 0 \cdot 1995; \\ & \log_{10} (1 \cdot 004) = 0 \cdot 0018; & \log_{10} (2 \cdot 783) = 0 \cdot 4445; \\ & \log_{10} (23 \cdot 13) = 1 \cdot 3642; & \log_{10} (9 \cdot 987) = 0 \cdot 9994; \\ & \log_{10} (1 \cdot 378) = 0 \cdot 1393; & \log_{10} (4 \cdot 677) = 0 \cdot 6700; \end{aligned}$$

Using these, find the values of—

$$\begin{aligned} \text{(i) } & \frac{27 \cdot 82}{3 \cdot 537} \quad \text{(ii) } \frac{4 \cdot 27 \times 8 \cdot 271}{5 \cdot 235} \quad \text{(iii) } \frac{84 \cdot 82}{52 \cdot 63 \times 1 \cdot 583} \quad \text{(iv) } \frac{2 \cdot 783 \times 23 \cdot 13}{9 \cdot 987 \times 1 \cdot 378} \end{aligned}$$

§ 6.6 (III) $\log_a (m^n) = n \log_a m$:

$\log_2 (3^5) = 5 \log_2 3$. Let us see whether this statement is true.

Suppose, $\log_2 3 = x$, then $2^x = 3$ (1)

$$\therefore 3^5 = (2^x)^5 = 2^{5x}$$

$$\therefore 2^{5x} = 3^5$$

From the definition of the logarithm, we get

$$5x = \log_2 (3^5) \quad \dots(2)$$

Substituting in (2) the value of x from (1), we get,

$$5 \times \log_2 3 = \log_2 (3^5)$$

$$\therefore \log_2 (3^5) = 5 \log_2 3$$

Similarly verify that this property holds good for different numbers also.

From this we can expect the following general result.

$$\log_a(m^n) = n \times \log_a m.$$

Proof : Suppose $\log_a m = x$, then $a^x = m$ (A)

$$\text{Now } (m^n) = (a^x)^n = a^{nx} \therefore a^{xn} = (m^n)$$

Converting this into logarithmic form, we get

$$nx = \log_a(m^n) \quad \dots(B)$$

Substituting in (B) the value of x from (A), we get $n \times \log_a m = \log_a(m^n)$

$$\therefore \log_a(m^n) = n \log_a m$$

Study the following examples based on this and also the previous results.

$$(1) \text{ Show that } 23 \log \left(\frac{10}{9} \right) - 6 \log \left(\frac{25}{24} \right) + 10 \log \left(\frac{81}{80} \right) = 1.$$

Here the base of the logarithm is 10.

Solution:

$$\begin{aligned} & 23 \log \left(\frac{10}{9} \right) - 6 \log \left(\frac{25}{24} \right) + 10 \log \left(\frac{81}{80} \right) \\ &= \log \left(\frac{10}{9} \right)^{23} - \log \left(\frac{25}{24} \right)^6 + \log \left(\frac{81}{80} \right)^{10} \\ &= \log \left[\frac{\left(\frac{10}{9} \right)^{23} \times \left(\frac{81}{80} \right)^{10}}{\left(\frac{25}{24} \right)^6} \right] \\ &= \log \left[\left(\frac{5 \times 2}{3^2} \right)^{23} \times \left(\frac{3^4}{5 \times 2^4} \right)^{10} \times \left(\frac{3 \times 2^3}{5^2} \right)^6 \right] \\ &= \log \left[\frac{5^{23} \times 2^{23}}{3^{46}} \times \frac{3^{40}}{5^{10} 2^{40}} \times \frac{3^6 \times 2^{18}}{5^{12}} \right] \\ &= \log (5^{23-10-12} \times 2^{23+18-40} \times 3^{40+6-46}) \\ &= \log (5^1 \times 2^1 \times 3^0) = \log_{10} 10 = 1 \end{aligned}$$

(2) Write the following example as the sum or difference of the logarithms of the numbers having no index/without index.

$$\log_a \left[\frac{\sqrt[5]{x \times y^3 \times z}}{\sqrt[3]{m} \times n^{\frac{2}{5}}} \right]$$

Solution :

$$\begin{aligned}
 \log_a \left[\frac{\sqrt[5]{x+y^2} \times z}{\sqrt[3]{m} \times n^{\frac{2}{5}}} \right] &= \log_a(\sqrt[5]{x+y^2} \times z) - \log_a(\sqrt[3]{m} \times n^{\frac{2}{5}}) \\
 &= \log_a[(x+y^2)^{\frac{1}{5}}] + \log_a z - (\log_a m^{\frac{1}{3}} + \log_a n^{\frac{2}{5}}) \\
 &= \log_a(x^{\frac{1}{5}} \times y^{\frac{2}{5}}) + \log_a z - \log_a m^{\frac{1}{3}} - \log_a n^{\frac{2}{5}} \\
 &= \log_a x^{\frac{1}{5}} + \log_a y^{\frac{2}{5}} + \log_a z - \frac{1}{3} \log_a m - \frac{2}{5} \log_a n \\
 &= \frac{1}{5} \log_a x + \frac{2}{5} \log_a y + \log_a z - \frac{1}{3} \log_a m - \frac{2}{5} \log_a n.
 \end{aligned}$$

$$\begin{aligned}
 (3) \log_{10} (3 \cdot 42) &= 0.5340 ; \log_{10} (5 \cdot 781) = 0.7620 \\
 \log_{10} (7 \cdot 8) &= 0.8921 ; \log_{10} (1 \cdot 237) = 0.2769 ; \log_{10} (1 \cdot 905) = 0.2800
 \end{aligned}$$

Using these find the value of $\frac{(3 \cdot 42)^2 \times \sqrt{5 \cdot 781}}{7 \cdot 8 \times (1 \cdot 237)^3}$.

Solution :

$$\begin{aligned}
 \log_{10} \left[\frac{(3 \cdot 42)^2 \times \sqrt{5 \cdot 781}}{7 \cdot 8 \times (1 \cdot 237)^3} \right] &= \log_{10}[(3 \cdot 42)^2 \times (5 \cdot 781)^{\frac{1}{2}}] - \log_{10}[7 \cdot 8 \times (1 \cdot 237)^3] \\
 &= \log_{10}(3 \cdot 42)^2 + \log_{10}(5 \cdot 781)^{\frac{1}{2}} - \log_{10}(7 \cdot 8) - \log_{10}(1 \cdot 237)^3 \\
 &= 2 \log_{10}(3 \cdot 42) + \frac{1}{2} \log_{10}(5 \cdot 781) - \log_{10}(7 \cdot 8) - 3 \log_{10}(1 \cdot 237) \\
 &= 2 \times (0.5340) + \frac{1}{2} \times (0.7620) - 0.8921 - 3 \times (0.2769) \\
 &= 1.0680 + 0.3810 - 0.8921 - 0.8307 \\
 &= 0.2800 \\
 &= \log_{10}(1 \cdot 905) \\
 \therefore \log_{10} \left[\frac{(3 \cdot 42)^2 \times \sqrt{5 \cdot 781}}{7 \cdot 8 \times (1 \cdot 237)^3} \right] &= \log_{10}(1 \cdot 905) \\
 \therefore \frac{(3 \cdot 42)^2 \times \sqrt{5 \cdot 781}}{7 \cdot 8 \times (1 \cdot 237)^3} &= 1 \cdot 905.
 \end{aligned}$$

EXERCISE 6.5

1. Simplify.

$$(i) 3 \log_a x - 2 \log_a y + 5 \log_a z \quad (ii) 3 \log_a 2 - 4 \log_a 4 + 2 \log_a 8$$

$$(iii) \frac{1}{4} \log_a 625 - \frac{1}{3} \log_a 125 + \frac{3}{2} \log_a 25$$

$$(iv) \frac{2}{3} \log_{10} 27 + \frac{3}{4} \log_{10} 81 - \frac{1}{2} \log_{10} 9$$

$$(v) \frac{1}{3} \log_5 (a+b)^3 + \frac{1}{2} \log_5 (a^2 - ab + b^2)^2.$$

2. Show that the following equalities are true.

$$(i) \log_{10} \left(\frac{77}{3} \right) - \log_{10} 180 + \log_{10} \left(\frac{27}{7} \right) - \log_{10} 55 = -2$$

$$(ii) 7 \log_{10} \left(\frac{10}{15} \right) + 5 \log_{10} \left(\frac{25}{24} \right) + 3 \log_{10} \left(\frac{81}{80} \right) = \log_{10} 2$$

$$(iii) 16 \log_{10} \left(\frac{10}{9} \right) - 4 \log_{10} \left(\frac{25}{24} \right) - 7 \log_{10} \left(\frac{80}{81} \right) = \log_{10} 5.$$

3. Write the following as the sum or difference of the logarithms of the numbers having no index.

$$(i) \log_a \left(\frac{x}{y} \right)^3 \quad (ii) \log_m (a^{\frac{1}{3}} \times b^{\frac{2}{3}}) \quad (iii) \log_k \left[\frac{x \times y^2}{z^3} \right]^{\frac{1}{5}}$$

$$(iv) \log_{10} [(3 \cdot 27)^2 \times (0 \cdot 38)^{\frac{1}{3}}] \quad (v) \log_{10} \left[\frac{(6 \cdot 45)^3 \times \sqrt[3]{0 \cdot 08}}{(7 \cdot 25)^{\frac{2}{3}} \times 4 \cdot 23} \right]$$

$$4. \log_{10} (2 \cdot 834) = 0 \cdot 4524; \log_{10} (22 \cdot 76) = 1 \cdot 3572; \log_{10} (7 \cdot 823) = 0 \cdot 8934;$$

$$\log_{10} (2 \cdot 797) = 0 \cdot 4467; \log_{10} (2 \cdot 782) = 0 \cdot 4443; \log_{10} (41 \cdot 21) = 1 \cdot 6150;$$

$$\log_{10} (23 \cdot 87) = 1 \cdot 3779; \log_{10} (1 \cdot 119) = 0 \cdot 0490.$$

Using these find the values of—

$$(i) (2 \cdot 834)^5 \quad (ii) (7 \cdot 823)^{\frac{1}{2}} \quad (iii) \frac{(2 \cdot 782)^2 \times \sqrt[3]{41 \cdot 21}}{23 \cdot 87}$$

§ 6.7 Common Logarithms :

$$342 \cdot 75 \times (0 \cdot 85)^7; \quad \frac{47 \cdot 4 \times 12 \cdot 7}{2 \cdot 8 \times 7 \cdot 12}; \quad \frac{\sqrt[3]{7 \cdot 5 \times 43} \times (71 \cdot 4)^2}{(3 \cdot 5)^2 \times 31 \cdot 4} \text{ etc.}$$

Calculations similar to those stated above and even more complicated ones occur frequently in different branches of mathematics and science. Such calculations are not only time-killing but they are also laborious. Let us see how the logarithms help us to simplify such calculations.

We can find logarithm of numbers to any base. The logarithms to the base 10 are convenient, hence in numerical calculations logarithms to the base 10 are used. This system of logarithms to the base 10 is called the system of **common logarithms**.

Finding $\log_{10} 100$ is the same as finding the index of 10 such that the power will be 100. Similarly finding $\log_{10} 4 \cdot 134$ is the same as finding the index of 10 such that the power will be $4 \cdot 134$. But this is not an easy job. To find the log of any number to the base 10, ready made tables are used. These tables are called **logarithm tables**. With the help of these tables we can find the logarithm of any number to the base 10.

From these tables we have $\log_{10} (4 \cdot 134) = 0 \cdot 6164$; $\log_{10} (52 \cdot 41) = 1 \cdot 7194 \dots \dots$ etc. As we shall consider logarithms to the base 10 only, we shall not mention the base. Whenever the base is not mentioned, it should be understood to be 10.

Now observe the logarithms of the numbers given below—

$$(i) \log (17 \cdot 35) = 1 \cdot 2392 \quad (ii) \log (3 \cdot 718) = 0 \cdot 5703$$

$$(iii) \log (947 \cdot 5) = 2 \cdot 9765 \quad (iv) \log (5378) = 3 \cdot 7306.$$

From the observation of the logarithms of the numbers, you will notice that the logarithm consists of two parts. The integral part in the logarithm is called the **characteristic** and the fractional part in the logarithm is called the **mantissa**,

e. g. (i) $\log(37.28) = 1.5714$. Here 1 is the characteristic and .5714 is the mantissa.

(ii) $\log(9.175) = 0.9626$. Here zero is the characteristic and .9626 is the mantissa.

It is clear from the above examples that

$$\text{Logarithm} = \text{Characteristic} + \text{Mantissa}$$

From the logarithm tables we get only the mantissa part. The characteristic is to be obtained by using some rules.

We must note that in the logarithm table the mantissa given is always positive.

Now, let us see how to find the characteristic of the logarithm of a given number.

§ 6.8 Characteristic :

When there is only one non-zero digit in the integral part of a number then let us call such number to be in the standard form. e. g. (i) 2.314, 7.812, 9.00, 8.53, 3, ... are numbers in the standard form. (ii) 21.73, 823.7, 0.082, 0.282 etc. are numbers which are not in the standard form. But these numbers can be reduced to the **standard form**.

$$\text{e. g. (i) } 21.73 = 2.173 \times 10^1$$

$$\text{(ii) } 823.7 = 8.237 \times 10^2$$

$$\text{(iii) } 0.082 = 8.2 \times 10^{-2}$$

$$\text{(iv) } 0.282 = 2.84 \times 10^{-1}$$

Now let us consider the characteristic of the logarithm of numbers in the standard form (i. e. numbers having one digit in the integral part). Let 3.542 be any number having one non-zero digit in the integral part.

It is obvious that $1 < 3.542 < 10$.

$$\therefore (10)^0 < 3.542 < (10)^1$$

$$\dots (\because a^0 = 1, a \neq 0)$$

Suppose $3.542 = (10)^x$. i.e. $x = \log_{10}(3.542)$

$$\therefore (10)^0 < (10)^x < (10)^1$$

$$\therefore 0 < x < 1 \quad \dots (\because \text{If } a > 0, x > 0, y > 0 \text{ and } a^x < a^y \text{ then } x < y)$$

This means that x is a positive number less than 1. As there is no integer between 0 and 1, the integral part of x is 0. But, x is the logarithm of 3.542. $\therefore$ Integral part in the logarithm of 3.542 is zero i.e. characteristic is zero and the non-integral part (i. e. mantissa) is a positive number less than 1. This is not only true for the number 3.542, but for any number which lies between 1 and 10 i.e. numbers having one non-zero digit in the integral part. Similarly by taking different numbers between 0 and 1 (i. e. numbers in the standard form) verify that the characteristic in their logarithm is zero.

Remember that the characteristic in the logarithm of a number in the standard form is zero.

e. g. 7.834, 5.3, 7, 1.003, 3, 1, 2.078 etc., the characteristic in the logarithm of these numbers is zero.

Using the result 'Characteristic, in the logarithm of a number in the standard form, is zero,' let us find the logarithms of other numbers.

$$\begin{aligned} \text{(i) } 33.48 &= 3.348 \times 10^1 \\ &= (10)^f \times (10)^1 && \dots 0 < f < 1. \\ &= (10)^{f+1} = (10)^{1+f} \end{aligned}$$

$\therefore$ Characteristic in the logarithm of 33.48 is 1.

Similarly verify that the characteristic in the logarithm of the numbers having two digits in the integral part, is one.

$$\begin{aligned} \text{(ii) } 432.22 &= 4.3222 \times (10)^2 \\ &= (10)^f \times (10)^2 && \dots 0 < f < 1. \\ &= (10)^{f+2} = (10)^{2+f} \end{aligned}$$

$\therefore$ Characteristic in the logarithm of 432.22 is 2.

Similarly verify that the characteristic comes out to be 2 in case of the numbers having three digits in the integral part.

$$\begin{aligned} \text{(iii) } 5872.81 &= 5.872281 \times 10^3 \\ &= (10)^f \times (10)^3 && \dots 0 < f < 1. \\ &= (10)^{f+3} = (10)^{3+f} \end{aligned}$$

$\therefore$ The characteristic in the logarithm of 5872.81 is 3.

Similarly verify that in case of the numbers having 4 digits in the integral part, the characteristic comes out to be 3.

$$\begin{aligned} \text{(iv) } 0.5837 &= 5.837 \times (10)^{-1} \\ &= (10)^f \times (10)^{-1} && \dots 0 < f < 1. \\ &= (10)^{f+(-1)} = (10)^{(-1)+f} \end{aligned}$$

$\therefore$ Characteristic in the logarithm of the number 0.5837 is -1 .

The negative characteristic is denoted by writing the minus sign on the top of that number instead of writing before that number. e. g. -1 is written as $\bar{1}$ and is read as 'bar 1'. If the minus sign is attached behind the characteristic, it will mean that the whole logarithm is negative. But the mantissa given in the tables is always positive. We want to indicate that only characteristic is negative, while mantissa is positive. To do so we write the negative sign of the characteristic at the top of it.

$$\begin{aligned} \text{(v) } 0.09827 &= 9.827 \times (10)^{-2} \\ &= (10)^f \times (10)^{-2} && \dots 0 < f < 1. \\ &= (10)^{f+(-2)} = (10)^{(-2)+f} = (10)^{\bar{2}+f} \end{aligned}$$

$\therefore$ The characteristic in the logarithm of 0.09827 is $\bar{2}$ (bar 2).

$$\begin{aligned} \text{(vi) } 0.0023 &= 2.3 \times (10)^{-3} \\ &= (10)^f \times (10)^{-3} && \dots 0 < f < 1 \\ &= (10)^{f+(-3)} = (10)^{(-3)+f} = (10)^{\bar{3}+f} \end{aligned}$$

$\therefore$ Characteristic in log of 0.0023 is $\bar{3}$ (bar 3).

Using the method indicated in the examples (iv), (v) and (vi) above find the characteristic in the logarithm of different numbers less than 1.

From the above examples you will notice that—

- (i) In case of the numbers greater than 1, the characteristic is non-negative.
- (ii) In case of the positive numbers less than 1, the characteristic is negative.

You have seen the method of finding the characteristic in the logarithm. Every time finding characteristic in this way is not suitable. Hence, let us try to find some thumb rule for this purpose. For this study the above examples and answer the following questions.

- (i) What is the relation between the number indicating the characteristic and the number through which you have to shift the decimal point to reduce the given number to its standard form.
- (ii) What is the correspondence between positive and negative characteristic and shifting of the decimal point to right or left, while reducing the number to its standard form.

From the answers to the above questions we have the following thumb rules to find the characteristic.

- (1) The numerical value of the characteristic is equal to the number of decimal places through which the decimal point has to be shifted while reducing a given number to its standard form.
- (2) The sign of the characteristic is positive, if the decimal point is to be shifted to the left while reducing the given number to its standard form.
- (3) The sign of the characteristic is negative, if the decimal point is to be shifted to the right, while converting the given number to its standard form.

Study the following examples in which the characteristic is obtained by using the above thumb rules.

(i) To reduce the number 23.724 to its standard form, we have to shift the decimal point to the left through 1 decimal place. $\therefore$ Characteristic is 1.

(ii) $\log(378.47)$ 3.7847 Characteristic = 2
 $\leftarrow$
 2

(iii) $\log(0.37847)$ 3.7847 Characteristic = $\overline{1}$
 $\rightarrow$
 1

While reducing 0.37847 to its standard form, we have to shift the decimal point to the right through 1 decimal place. $\therefore$ Characteristic is $\overline{1}$

(iv) $\log(3.7847)$ 3.7847 $\therefore$ Characteristic = 0
 $\rightarrow$
 0

(v) $\log(0.037847)$ 03.7847 $\therefore$ Characteristic = $\overline{2}$
 $\rightarrow$
 2

(vi) $\log (37.847)$	3.7847 $\leftarrow$ 1	$\therefore$ Characteristic = 1
(vii) $\log (58732)$	5.8732 $\leftarrow$ 4	$\therefore$ Characteristic = 4
(viii) $\log (0.00873)$	008.73 $\overline{\leftarrow}$ 3	$\therefore$ Characteristic = $\overline{3}$
(ix) $\log (53)$	5.3 $\leftarrow$ 1	$\therefore$ Characteristic = 1
(x) $\log (4327.8)$	4.3278 $\leftarrow$ 3	$\therefore$ Characteristic = 3
(xi) $\log (0.0000125)$	00001.25 $\overline{\leftarrow}$ 5	$\therefore$ Characteristic = $\overline{5}$

From the above examples, you must have noticed that—

(i) In case of numbers greater than 1, the characteristic is equal to the number of digits in the integral part minus one. (ii) In case of numbers less than 1, the characteristic is negative and the numerical value of the characteristic is equal to the number of successive zeros immediately after the decimal point plus one.

EXERCISE 6-6

1. Write the characteristics of the logarithm of the following numbers.

- | | | | |
|----------------|--------------|---------------|------------------------|
| (i) 32.57 | (ii) 0.0342 | (iii) 5.007 | (iv) 0.009901 |
| (v) 0.1001 | (vi) 782.1 | (vii) 3.298 | (viii) 0.000428 |
| (ix) 287.0021 | (x) 5372 | (xi) 8.0012 | (xii) 0.1348 |
| (xiii) 23.76 | (xiv) 2.8199 | (xv) 0.008427 | (xvi) 285.31 |
| (xvii) 0.08219 | (xviii) 475 | (xix) 52 | (xx) 15.7 (xxi) 7.0081 |

2. 2701, 23, 1010, 80, 27895.

Place the decimal point in the proper place so that the characteristics in the above numbers so obtained will be 1, $\overline{2}$, 0, 2, $\overline{1}$, 4, $\overline{3}$, 3, $\overline{4}$.

§ 6.9 Logarithm Tables :

You know that $\text{Logarithm} = \text{Characteristic} + \text{Mantissa}$. You also know how to determine the characteristic part of the logarithm of a given number. Now you will know how to determine the mantissa part of the logarithm of a number.

Mantissa is obtained from the logarithm tables. Our logarithm tables can give the mantissa part of the numbers upto four digits only. Therefore they are called

four figure log tables. There are 5 figure, 7 figure and even 20 figure log tables. If there are more than 4 digits in a number then reduce that number to four digit significant number.

e.g. (i) 3.43216 is reduced to four digit significant number as 3.432.

(ii) The number 86.765, when reduced to four digit significant number, becomes 86.77.

Now open the page of the log table with title 'Logarithms' and observe the following points.

LOGARITHMS																		
A		B										C						
		0	1	2	3	4	5	6	7	8	9	Mean Difference						
												1	2	3	4	5	6	7
10		0000	0043							0378								
11																		
12																		
13																		

(i) The first column contains the numbers from 10 to 99. Let us call this column as A. (ii) The first row at the top consists of numbers from 0 to 9. Let us call this part of the log page as B. Corresponding to each of the numbers in B, there is a column in which 4 digit numbers are written. (iii) Next to B, there is a row which consists of numbers from 1 to 9. Let us call this part of the table as C. Corresponding to each of the numbers in C, there is a column in which some numbers are written.

Now let us see how to find the mantissa part of the logarithm from this table.

Note—(i) From the chart given on pages no. 100, 101, with the help of the examples study the method explained in each step and then repeat the same with the help of the log table with you.

(ii) As per procedure in the chart first study example (1) and then the next.

EXERCISE 6.7

Using log table, find the logarithm of the following numbers—

1. 2.431
2. 15.17
3. 0.8156
4. 0.003842
5. 12
6. 34.5
7. 5
8. 0.084723
9. 0.00273123
10. 437892
11. 99.01
12. 0.09103

13. 42.71	14. 4.271	15. 0.4271	16. 0.04271
17. 0.004271	18. 427.1	19. 91.93	20. 0.09193
21. 0.2387	22. 2.387	23. 543768	24. 8734123
25. 8	26. 83	27. 9.72	28. 30.01
29. 2000	30. 20.		

§ 6.10 Antilogarithm:

What is the logarithm of 1000 to the base 10? To find this we try to obtain a number such that 10 raised to that number gives 1000. Now let us consider the converse of this problem.

To find the number that we get when the index of the base 10 is 3 means finding antilog of 3 when the base is 10. From this it is clear that finding log and antilog are mutually reverse processes.

- (i) $\log_{10} 100 = 2$ $\therefore$ Antilog 2 = 100
- (ii) $\log_{10} 1000 = 3$ $\therefore$ Antilog 3 = 1000
- (iii) $\log_{10} (17.22) = 1.2360$ $\therefore$ Antilog (1.2360) = 17.22
- (iv) $\log_{10} (6.537) = 0.8154$ $\therefore$ Antilog (0.8154) = 6.537
- (v) $\log_{10} (0.0048) = \overline{3}.6812$ $\therefore$ Antilog ($\overline{3}.6812$) = 0.0048
- (vi) $\log_{10} (0.05321) = \overline{2}.7260$ $\therefore$ Antilog ($\overline{2}.7260$) = 0.05321
- (vii) $\log_{10} (432.81) = 2.6363$ $\therefore$ Antilog (2.6363) = 432.81
- (viii) $\log_{10} (43.281) = 1.6363$ $\therefore$ Antilog (1.6363) = 43.281

Now let us see how to find the antilogarithm of a given number.

Note that the number whose antilog we are finding is the logarithm of some number. Hence the integral part in it is its characteristic and the non-integral part is the mantissa. While finding the antilog from the table we do not consider the characteristic part and concentrate our attention on the mantissa part alone. So that in effect we are considering only the decimal part of the given number whose antilog we wish to find, i.e. the four digits (including zeros in the beginning if they occur) to the right of decimal point are only considered. e.g. while finding antilog of 7.8724 from the antilog table we consider only .8724. By using the same method used to find mantissa, we can find the antilog from antilog table of the four digit number which occurs after the decimal point. In the logarithm table book, the antilogarithm table is given after the logarithm tables.

After finding the antilog, the decimal point is to be placed in such a way that the characteristic in the logarithm of the number so obtained should be equal to the integral part in the given number.

PROCEDURE																																																																																																																																																																		
	Example (1)	Example (2)																																																																																																																																																																
1	38.1292	0.043259																																																																																																																																																																
2	381292	043259																																																																																																																																																																
3	3813	4326																																																																																																																																																																
4	(i) Note in section A of log table the first 2 digits of the number obtained in step (3). (ii) Note (Mark) in the first row of section B of log table the third digit of the number obtained in step (3).	<table><tr><td colspan="10">B</td><td colspan="10">C</td></tr><tr><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td></tr><tr><td>10</td><td>11</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>10</td><td>11</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td></tr><tr><td>37</td><td>38</td><td>39</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>42</td><td>43</td><td>44</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td></tr></table> <table><tr><td colspan="10">B</td><td colspan="10">C</td></tr><tr><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td></tr><tr><td>10</td><td>11</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>10</td><td>11</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td></tr><tr><td>37</td><td>38</td><td>39</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>42</td><td>43</td><td>44</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td></tr></table>	B										C										0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	10	11	-	-	-	-	-	-	-	-	10	11	-	-	-	-	-	-	-	-	37	38	39	-	-	-	-	-	-	-	42	43	44	-	-	-	-	-	-	-	B										C										0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	10	11	-	-	-	-	-	-	-	-	10	11	-	-	-	-	-	-	-	-	37	38	39	-	-	-	-	-	-	-	42	43	44	-	-	-	-	-	-	-
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PROCEDURE

Example (1)

Example (2)

Step No.	Method of finding logarithm from table	Example (1)	Example (2)
6	(i) Note in section C of the log table the fourth digit of the number in step (3). (ii) Observe and write the number where the row corresponding to number of the section A in the step 4 intersects the column corresponding to the number in section C.		
7	Add the number obtained in step (6) to the number obtained in step (5). This is the mantissa of the logarithm of the given number.	$\begin{array}{r} 0.5809 \\ + \quad 3 \\ \hline 0.5812 \end{array}$	$\begin{array}{r} 0.6355 \\ + \quad 6 \\ \hline 0.6361 \end{array}$
8	Write the characteristic in the logarithm of the given number.	1	2
9	Logarithm of a number = Characteristic + Mantissa. Using this, from steps 7 and 8 write the log of the given number.	$\log (38.1280) = 1.5812$	$\log (0.0432591) = \bar{2}.6361$

Study the following illustrations. Find the antilog of the following numbers.

(i) 1.7824

(ii) 1.2384

(iii) 0.0024.

Step No.	Method of finding antilog	Example (1)	Example (2)	Example (3)
1	Write the numbers whose antilog we want to find.	1.7824	$\overline{1}.2384$	0.0024
2	Write the digits which occur to the right of the decimal point.	.7824	.2384	.0024
3	Find the antilog of the numbers in step (2) by using the same procedure which is used to find mantissa.	6059	1732	1006
	Place the decimal point in the number obtained in step (3) in such a way that the characteristic in the logarithm of the number so obtained is equal to the integral part in the given number.	60.59	0.1732	1.006

EXERCISE 6.8

Using antilog table, find the antilogarithms of the following numbers.

1. 1.9406 2. 0.0026 3. $\overline{2}.4371$ 4. 0.7123 5. $\overline{1}.2748$ 6. 2.3348
 7. 0.3348 8. $\overline{1}.3348$ 9. 1.3348 10. $\overline{2}.3348$ 11. 3.2341 12. 1.8732
 13. 0.0005 14. $\overline{3}.0872$ 15. 0.0001 16. 0 17. 1.0101 18. 3.4827

§ 6.11 If $x > y$ then $\log_a x > \log_a y$. $x, y > 0$ and $a > 1$.

(1) We know that $5 > 3$. Then, is $\log_{10} 5$ also greater than $\log_{10} 3$? Let us see whether this statement is true.

Let $\log_{10} 5 = m$ and $\log_{10} 3 = n$, then we have $(10)^m = 5$ and $(10)^n = 3$.

But $5 > 3$. $\therefore (10)^m > (10)^n$.

Using the result, 'If $a > 1$ and $a^p > a^q$ then $p > q$,' we get, $m > n$.

But $m = \log_{10} 5$ and $n = \log_{10} 3$ $\therefore \log_{10} 5 > \log_{10} 3$

Similarly you verify that $\log_{10} 9 > \log_{10} 7$, $\log_5 3 > \log_5 2$, $\log_3 7 > \log_3 4$ etc.

From these examples we arrive at the general result as follows—

If $x, y > 0$, $a > 1$ and $x > y$ then $\log_a x > \log_a y$.

Let us prove this result.

Let us suppose, $\log_a x = m$ and $\log_a y = n$. then $x = a^m$ and $y = a^n$

But $x > y$. $\therefore a^m > a^n$.

Using the result, 'If $x, y > 0, a > 1$ and $a^p > a^q$ then $p > q$ ', we get
 $\therefore a^m > a^n \quad \therefore m > n$

But $m = \log_a x$ and $n = \log_a y$.
 $\therefore \log_a x > \log_a y$.

(2) Is the converse of the above statement true?

On the same lines of the proof of the above theorem, you verify that the converse of the above statement is true, i.e. verify that :

If $x, y > 0, a > 1$ and $\log_a x > \log_a y$ then $x > y$.

Study the following solved examples. Using log tables find the values of—

1. 0.03143×17.84
2. $\frac{2.1327}{0.1271}$
3. $(3.412)^5$
4. $\sqrt[3]{0.7834}$
5. $\frac{(2.312)^2 \times \sqrt[4]{70.33}}{7.813 \times 0.0872}$

Solution :

$$\begin{aligned} 1. \log (0.03143 \times 17.84) \\ &= \log (0.03143) + \log (17.84) \\ &= \bar{2}.4973 + 1.2514 = (\bar{2} + 1) + (.4973 + .2514) \\ &= \bar{1} + (.7487) = \bar{1}.7487. \end{aligned}$$

$$\begin{aligned} \text{Antilog} [\log (0.03143 \times 17.84)] &= \text{Antilog } \bar{1}.7487 \\ \therefore 0.03143 \times 17.84 &= 0.5607 \end{aligned}$$

$$\begin{aligned} 2. \log \left(\frac{2.1327}{0.1279} \right) &= \log (2.1327) - \log (0.1271) \\ &= 0.3290 - \bar{1}.1370 = 1.1920 \end{aligned}$$

$$\therefore \frac{2.1327}{0.1279} = \text{Antilog} (1.1920) = 15.56$$

$$\begin{aligned} 3. \log (3.412)^5 &= 5 \log (3.412) = 5 \times 0.5331 = 2.6655 \\ \therefore (3.412)^5 &= \text{Antilog} (2.6655) = 462.9 \end{aligned}$$

$$\begin{aligned} 4. \log (\sqrt[3]{0.7834}) &= \log (0.7834)^{\frac{1}{3}} = \frac{1}{3} \log (0.7834) = \frac{1}{3} \times (\bar{1}.8940) \\ &= \frac{\bar{1}}{3} + \frac{0.8940}{3} = \frac{\bar{3} + 2}{3} + \frac{.8940}{3} = \frac{\bar{3} + 2.8940}{3} = \frac{\bar{3}}{3} + \frac{2.8940}{3} \\ &= \bar{1} + 0.9647 = \bar{1}.9647 \\ \therefore (0.7834)^{\frac{1}{3}} &= \text{Antilog} (\bar{1}.9647) = 0.9219. \end{aligned}$$

$$\begin{aligned}
5. \quad & \log \left[\frac{(2.312)^3 \times \sqrt[4]{70.33}}{7.813 \times 0.0872} \right] \\
&= \log \left[(2.312)^3 \times (70.33)^{\frac{1}{4}} \right] - \log [7.813 \times 0.0872] \\
&= \log (2.312)^3 + \log (70.33)^{\frac{1}{4}} - \log (7.813) - \log (0.0872) \\
&= 2 \log (2.312) + \frac{1}{4} \log (70.33) - [\log (7.813) + \log (0.0872)] \\
&= 2 \times 0.3640 + \frac{1}{4} \times 1.8472 - (0.8929 + 2.9405) \\
&= 0.7280 + 0.4618 - 1.8334 \\
&= 1.1898 - 1.8334 = 1.3564 \\
\therefore & \frac{(2.312)^3 \times \sqrt[4]{70.33}}{7.813 \times 0.0872} = \text{Antilog } (1.3564) = 22.70
\end{aligned}$$

EXERCISE 6.9

Using log table find the values of—

1. 44.8×21.9
2. $\frac{703.9}{22.91}$
3. $(3.241)^3$
4. $\sqrt[4]{17.82}$
5. $2.8 + 9.33 \times 8.437$
6. $\frac{27.41 \times 8.27}{97.98}$
7. $\frac{33.432}{13.83 \times 17.5}$
8. $(0.0987)^4$
9. $\sqrt[3]{0.08271}$
10. $\sqrt{0.8351}$
11. $\frac{4.732 \times (0.1785)^2}{\sqrt{276.9}}$
12. $\frac{\sqrt[3]{8.34} \times (3.12)^2}{(4.312)^3 \times \sqrt{24.3}}$
13. $\sqrt[4]{2.709} \times \sqrt[3]{1.2387}$
14. $\frac{(0.439)^{\frac{3}{2}} \times (14.7)^2}{(0.0062)^{\frac{1}{4}}}$
15. $\frac{(6.45)^3 \times \sqrt[3]{0.00034}}{(9.37)^2 \times \sqrt[4]{8.93}}$

Chapter 7 : POLYNOMIALS

You have already learnt about open phrases. Open phrases are also expressions. Some of them are called polynomials.

Observe the following expressions.

$$1. 3x^2 - 2x + 7 \quad 2. \frac{1}{2}x^3 + \frac{3}{5}x^2 - 4x + \frac{7}{6} \quad 3. 8y^4 - \sqrt{7}y^3 + \frac{5}{9}y^2 + 4$$

$$4. -\frac{3}{4}x^5 + 9x^4 + \frac{4}{3}x + \frac{6}{x} - \frac{5}{x^2} \quad 5. 24x^2 - 7\sqrt{x} + 5\sqrt[3]{x} + 2$$

In forming the above expressions, real numbers and symbols x and y are used. The indices of x and y in the first three expressions are positive integers, while those in the fourth and fifth are negative integers and even rational numbers that are not integers.

The first three of the above expressions in which the indices of x and y are positive integers, are polynomials. The fourth and the fifth expressions are not polynomials. For the present we intend to study only polynomials.

We shall assume that the symbols x and y , used in polynomials have properties almost similar to those of the numbers.

(1) The sum of x and 7 is written as $x + 7$.

The sum of x and 0 is $x + 0 = x$.

(2) We know that $5x$ means $x + x + x + x + x$ but the same meaning cannot be given to $\frac{5}{3}x$ (the product of $\frac{5}{3}$ and x).

(3) $1 \times x = x$; $(-1)x = -x$; $0 \times x = 0$.

Also $x \times x = x^2$; $x^3 \times x^2 = x^{3+2} = x^5$.

(4) Addition and multiplication of the symbol x and any real number are assumed to obey the commutative, associative and distributive laws.

Let us multiply $5x$ and $3x$.

$$\begin{aligned} (5x) \times (3x) &= (5 \times x) \times (3 \times x) \\ &= [(5 \times x) \times 3] \times x \\ &= [5 \times (x \times 3)] \times x \\ &= [5 \times (3 \times x)] \times x \\ &= [(5 \times 3) \times x] \times x \\ &= (5 \times 3) \times (x \times x) \\ &= 15 \times x^2 \\ &= 15x^2 \end{aligned}$$

.. Meaning of $5x$ and $3x$

.. Asso. of Multi.

.. Asso. of Multi.

.. Comm. of Multi.

.. Asso. of Multi.

.. Asso. of Multi.

.. $x \times x = x^2$

Let us add $5x$ and $3x$.

$$\begin{aligned} 5x + 3x &= 5 \times x + 3 \times x \\ &= (5 + 3) \times x \\ &= 8 \times x = 8x \end{aligned}$$

.... Meaning of $5x$ and $3x$
.... Distributivity.

Similarly $7x + \frac{4}{3}x = 7 \times x + \frac{4}{3} \times x$

$$= \left(7 + \frac{4}{3}\right) \times x = \frac{25}{3} \times x = \frac{25}{3}x$$

Note that if a polynomial in x or y or z contains $a, b, c, \dots$ then $a, b, c, \dots$ will be taken to represent real numbers.

Ex. 1 : If $ax + b = 3x + 4$, then $a = 3$ and $b = 4$

Ex. 2 : If $ax^2 + 3x + c = 2x^2 + bx - 5$, then $a = 2, b = 3$ and $c = -5$.

§ 7.1 Monomials:

If n is a positive integer, we know that x^n is called the n th power of x . If a is any real number, then ax^n is called a **monomial** in x . Observe that a monomial consists only one term.

The following are monomials—

$$7x^4; -5x^2; \frac{1}{8}x^{12}; -\frac{4}{5}x; 205x^7$$

We also know that $x^0 = 1$ i.e. $1 = x^0$. Therefore we can write $7 = 7 \times 1 = 7x^0$. Such monomials are called **constant monomials**. $3; -\frac{5}{2}; 0.12; -200; \sqrt[3]{5}$ are monomials. The expression 0 is also a constant monomial.

Definitions :

- (1) If $a \neq 0$ and n is a positive integer then ax^n is called a **monomial** of degree n . (The degree of this monomial is n).
- (2) If c is any non-zero real number, then it can be written as cx^0 . Therefore the degree of a non-zero constant monomial c is zero.
- (3) The monomial 0 can be written in any of the forms $0x^2; 0x^3; 0x^n$. Hence it is not possible to determine its degree. Therefore we do not assign any degree to the polynomial ' 0 '.
- (4) If ax^n is a monomial, then a is called its **coefficient**. The coefficient of the constant monomial c is c itself.

Ex. 1 : $3x^2$ is a monomial in x . Its degree is 2 and its coefficient is 3.

Ex. 2 : $-\frac{1}{7}y$ is a monomial in y . Its degree is 1 and its coefficient is $-\frac{1}{7}$.

Ex. 3 : 53 is a non-zero constant monomial. Its degree is zero and its coefficient is 53.

EXERCISE 7.1

1. State the degree and the coefficient of each of the following monomials.

- (i) $4x^3$ (ii) $-7x^5$ (iii) $\frac{9}{13}x^{12}$ (iv) $-72x$
(v) 100 (vi) $mx^n; (m \neq 0, n \in \mathbb{N})$ (vii) $2167x^{1000}$ (viii) 0 .

2. Write one monomial each of degree 5, 3, 1, 0.

3. Write one monomial each having coefficient 7, $\frac{4}{11}$, -5 , 1, 0.

4. Fill in the blanks.

- (i) The degree of the monomial $-5x^3$ is — and its coefficient is —.
- (ii) The degree of a monomial is a — number.
- (iii) The degree of a non-zero constant monomial is —.
- (iv) The monomial — has — degree.
- (v) The coefficient of the monomial x^3 is —.
- (vi) The coefficient of the monomial $-x^3$ is —.
- (vii) The index of x in a non-zero constant monomial in x is —.
- (viii) Of $5x^2$ and $5x^{-2}$, — is not a monomial, because —.
- (ix) Of $5x^2$ and $5\sqrt{x}$, — is not a monomial, because —.
- (x) Of $3x$ and $\frac{3}{x}$, — is not a monomial, because —.

§ 7.2 Polynomials :

Now let us study polynomials a little further. From the word 'polynomial', you would naturally think that a polynomial should contain more than one term. But monomials are also considered as polynomials i.e., monomial is a special type of polynomial. A polynomial is expressed in the form of a sum of monomials of different degrees. A monomial however can contain two or more symbols e.g. x^2y^3 ; $4x^2yz^4$ etc. But, we are going to study polynomials containing only one symbol.

Consider the following polynomials.

- (1) $4x^3$ (2) -17 (3) $7x^2 + 5$ (4) $4y - 7y^3 + 11y^2 - \frac{5}{2}y^4 + 9$
- (5) $-\frac{1}{3}x^2 + \frac{2}{7}x^5 - 5 + \frac{3}{8}x^3$.

The first is a monomial in x and its degree is 3.

The second is a constant monomial and its degree is zero. This is called a **constant polynomial**.

The third is a polynomial in x and it is expressed as a sum of two monomials $7x^2$ and 5. Of these the degree of the monomial $7x^2$ is 2 and it is the maximum degree of the monomials in this polynomial.

The fourth is a polynomial in y . The degree of the monomial $-\frac{5}{2}y^4$ is 4 which is the maximum degree of the monomials in this polynomial.

The fifth is a polynomial in x . The degree of the monomial $\frac{2}{7}x^5$ is 5 which is the maximum degree of the monomials in this polynomial.

Thus a polynomial can be considered as a sum of monomials of different degrees.

The monomials whose sum forms a polynomial are called the **terms** of the polynomial.

The highest of the degrees of the terms of a polynomial is called the **degree** of that polynomial.

The degree of any term of a polynomial has always to be a whole number (i.e. a non-negative integer). It can never be a negative integer or a non-integral rational number. But the indices of the terms of an 'expression' can be negative integers or

even rational numbers. This distinguishes the polynomials from other expressions. Therefore

$$4x^3 - 7x^2 + 5 + \frac{2}{x} - \frac{3}{x^2}$$

$$9x^2 + 11x - 5\sqrt{x} + \sqrt[3]{x^2}$$

are expressions but not polynomials.

It is convenient to write the polynomials in the descending order of the indices of their terms. Thus we can write the fourth and the fifth polynomials given on page 107 as follows.

$$-\frac{5}{2}y^4 - 7y^3 + 11y^2 + 4y + 9 ; \quad \frac{2}{7}x^5 + \frac{4}{3}x^3 - \frac{1}{3}x^2 - 5$$

Now let us consider the polynomial : $5x - 3x^8 + 4x^2 - 7x^5 + 2$.

This can be written in the descending order of the indices of its terms as follows.

$$-3x^8 - 7x^5 + 4x^2 + 5x + 2 \dots \dots \dots (1)$$

Do you observe that some powers of x are missing ? Which are they ? The terms containing the powers x^7, x^6, x^4, x^3 are absent. This means that the coefficient of each of these terms is zero. Now inserting these missing terms (taking their coefficients to be zero), in the descending order of the indices, it will look like—

$$-3x^8 + 0x^7 + 0x^6 - 7x^5 + 0x^4 + 0x^3 + 4x^2 + 5x + 2.$$

There are 9 terms and the degree of the polynomial is 8. Thus a polynomial of degree 8, when written in full (including those terms with zero coefficients) will contain 9 terms. The coefficients of these 9 terms written in order are

$$(-3, 0, 0, -7, 0, 0, 4, 5, 2) \dots \dots \dots (2)$$

These are called the **coefficients** of the polynomial.

We can obtain all the terms of a polynomial if we just know all its coefficients (including the zero coefficients). Therefore, by writing all the coefficients of the terms in the descending order of their indices, we can express a polynomial without explicit use of the symbols x, y etc. When a polynomial is expressed thus, the form, we get, is known as **detached coefficients form**. But in short, we shall call it **coefficient form**. Thus (2) above is the coefficient form of the polynomial given in (1). Note that, when this form is used, the zero coefficients must be shown. If a polynomial is expressed, by writing the powers of the symbols x, y , etc. then the form so obtained, is known as the **index form** of a polynomial. Thus (1) is the index form of a polynomial and (2) is its coefficient form.

Let us consider the following polynomial written in the index form.

$$4x^8 - 7x^5 + 5x^4 + 3x^2 - 6.$$

Inserting the missing terms, this polynomial can be written as

$$4x^8 + 0x^7 + 0x^6 - 7x^5 + 5x^4 + 0x^3 + 3x^2 + 0x - 6.$$

The coefficients of this polynomial in order are

$$(4, 0, 0, -7, 5, 0, 3, 0, -6)$$

This is the coefficient form of the given polynomial.

Now, if a polynomial is given in the index form, you can write it in the coefficient form and if it is given in the coefficient form, you can express it in the index form.

One thing to be noted here, is that the degree of a polynomial is 1 less than the number of all of its terms (including those with zero coefficients).

Ex. 1: Express the following polynomial, given in the coefficient form in the index form. $(5, 0, -4, -3, 0, 0, -9)$

Here the number of coefficients is 7

$\therefore$ the degree of the polynomial is 6.

$\therefore 5x^6 + 0x^5 - 4x^4 - 3x^3 + 0x^2 + 0x - 9$ is the given polynomial in x .

Deleting now, the terms with zero coefficients, this becomes

$$5x^6 - 4x^4 - 3x^3 - 9.$$

This is the required index form of the given polynomial.

Note: There is nothing special about the symbol x . We can as well use y or z or any other suitable symbol. Hence while changing the coefficient form of a polynomial to the index form, you are free to choose any symbol you like.

Ex. 2: Express the polynomial x^3 in the coefficient form.

$$x^3 = 1x^3 + 0x^2 + 0x + 0$$

$\therefore (1, 0, 0, 0)$ is the required coefficient form.

Ex. 3: Express the polynomial ax^4 ($a \neq 0$) in the coefficient form.

$$ax^4 = ax^4 + 0x^3 + 0x^2 + 0x + 0.$$

$\therefore (a, 0, 0, 0, 0)$ is the required coefficient form.

Note: (1) What are the coefficients of (i) x^3 ; (ii) $-x$ in the polynomial $x^3 + 6x^2 - x + 2$?

$$x^3 = 1 \cdot x^3 \therefore \text{coefficient of } x^3 \text{ is } 1.$$

$$-x = (-1) \cdot x \therefore \text{coefficient of } -x \text{ is } -1.$$

(2) In the above polynomial the last term 2, can be written as $2 = 2x^0$. Hence the degree of this term is 0. This term is called the constant term of the given polynomial. This term does not contain x . Such a term may even be zero. In a polynomial, a term which does not contain x (i.e. any symbol x, y or z etc.) is called a **constant term**.

EXERCISE 7.2

1. State the degrees of the following polynomials and express them in the coefficient form.

$$(i) 3x^2 - 4x + 5 \quad (ii) 6x^3 - 5x^2 + 7x + 3 \quad (iii) 7x^4 - 3x^2 + 2x$$

$$(iv) -x^7 + 5x^4 - 3x^2 \quad (v) 3x + 4 \quad (vi) -5x + 9 \quad (vii) 7x \quad (viii) 13$$

$$(ix) x^4 \quad (x) x^3 - 1 \quad (xi) -5 \quad (xii) -3x^4 + 4 \quad (xiii) 1 \quad (xiv) x$$

$$(xv) ax^4 - x^3 + 2x - b \quad (xvi) -x^7 + px^6 - 7x^4 + qx - 1.$$

2. State the degrees of the following polynomials and express them in the index form.

$$(i) (3, 5, 1) \quad (ii) (7, -2, 4) \quad (iii) (1, 1) \quad (iv) (1, 0) \quad (v) (1, 0, 0) \quad (vi) (-3, 0, 0, -3)$$

$$(vii) (5) \quad (viii) \left(-\frac{2}{3}, \frac{1}{2}, 0, \frac{4}{7}, 0, 0, -\frac{5}{8}\right) \quad (ix) (3, 0, 0, 0) \quad (x) (a, b, c, d, e)$$

$$(xi) (a, 1, -2, 0, b) \quad (xii) (1, p, 0, -4, 0, q, -1)$$

3. If $ax^4 + bx^3 + cx^2 + dx + e$ is the polynomial $5x^4 + 3x^2 - 7x + 8$, what are a, b, c, d, e ?

§ 7.3 Addition and Subtraction of Polynomials:

Terms in the same symbol and of the same degree are called like terms. Two like terms can be added by using distributive law e. g., $3x^4$ and $7x^4$ are like terms.

$$\begin{aligned} 3x^4 + 7x^4 &= 3 \times x^4 + 7 \times x^4 \\ &= (3 + 7) \times x^4 \quad \dots (\text{Distributivity}) \\ &= 10 \times x^4 = 10x^4 \end{aligned}$$

By taking together the like terms and adding them as above, we can find the sum of two polynomials in the same symbol. This will be clear from the following examples.

Ex. 1: Add $2x^2 + 5x - 3$ and $4x^2 - 7x + 8$.

$$\begin{aligned} \text{Required sum} &= (2x^2 + 5x - 3) + (4x^2 - 7x + 8) \\ &= (2x^2 + 4x^2) + (5x - 7x) + (-3 + 8) \\ &= 6x^2 - 2x + 5. \end{aligned}$$

Ex. 2: Add $4x^3 - 7x^2 + 2x + 1$ and $-5x^2 + ax - 4$.

$$\begin{aligned} \text{Required sum} &= (4x^3 - 7x^2 + 2x + 1) + (-5x^2 + ax - 4) \\ &= (4x^3) + (-7x^2 - 5x^2) + (2x + ax) + (1 - 4). \\ &= 4x^3 - 12x^2 + (2 + a)x - 3. \end{aligned}$$

Ex. 3: Add $ax^3 + 8x^2 - px$ and $4x^3 + bx^2 - qx + 8$.

$$\begin{aligned} \text{Required sum} &= (ax^3 + 8x^2 - px) + (4x^3 + bx^2 - qx + 8) \\ &= (ax^3 + 4x^3) + (8x^2 + bx^2) + (-px - qx) + (8) \\ &= (a + 4)x^3 + (8 + b)x^2 - (p + q)x + 8. \end{aligned}$$

Some times it is convenient to perform addition of polynomials, by writing like terms of the polynomials one below the other. Study the following example.

Ex. 4: Add $3x^3 + 5x^2 + 3$; $-x^3 - 7x^2 - 2x + 6$ and $-2x^3 + x^2 + 8x - 7$.

$$\begin{array}{r} 3x^3 + 5x^2 + 0x + 3 \\ + \quad -x^3 - 7x^2 - 2x + 6 \\ + \quad -2x^3 + x^2 + 8x - 7 \\ \hline -x^2 + 6x + 2. \end{array}$$

Addition of polynomials can also be carried out by using the coefficient form. The procedure will be clear from the following solved examples.

Ex. 5: Add $4x^3 - 3x^2 + 7x - 9$ and $7x^3 - 4x + 15$.

Let us express these polynomials in the coefficient form and write them one below the other in such a way that the coefficients of like terms appear one below the other in the same column. Adding the corresponding coefficients column-wise, we get a polynomial in the coefficient form. The polynomial so obtained, is the required sum.

$$\begin{array}{rrrr} 4 & -3 & 7 & -9 \\ + & 7 & 0 & -4 & 15 \\ \hline 11 & -3 & 3 & 6 \end{array}$$

The number of coefficients in this polynomial is 4.

$\therefore$ The degree of this polynomial is 3.

$\therefore$ Required sum $= 11x^3 - 3x^2 + 3x + 6$.

Ex. 6: Add $ax^4 + 7x^3 - 12x + p$ and $5x^4 + qx^2 + 5x - r$.

$$\begin{array}{rcccccc}
 & a & 7 & 0 & -12 & p \\
 + & 5 & 0 & q & 5 & -r \\
 \hline
 & (a+5) & 7 & q & -7 & (p-r)
 \end{array}$$

The number of coefficients in this polynomial is 5.

$\therefore$ The degree of this polynomial is 4.

$\therefore$ Required sum $= (a+5)x^4 + 7x^3 + qx^2 - 7x + (p-r)$.

EXERCISE 7.3

Add the following polynomials and state the degrees of their sums. Also add them by writing them in detached coefficients form and check your answers.

1. $3x^3 + 4x + 7$; $5x^2 - 4x + 8$
2. $7x^2 + 12$; $-6x^2 + 9x - 12$
3. $x^3 + 2x$; $x^3 - 3x$
4. $x^4 + x^2 + 2$; $x^2 + 2x + 1$
5. $x^4 + 3x^2 - 7$; $x^5 - x^4 - 3x^2 + 7$.
6. $5x^3 - 2$; $x^2 + 2x + 4$; $-3 + 5x - 7x^3 - 9x^2$.
7. $7x^2 + 5x - 9$; 17.
8. $5x^2 - 3x + 4$; -15.
9. $3x^2 + 2x - 7$; 0.
10. $3x^2 - 4x + 5$; $-3x^2 + 4x - 5$.
11. $5x - 7x^2 + 4$; $7x^2 - 5x - 4$.
12. $ax^3 + bx^2 + cx + d$; $-ax^3 - bx^2 - cx - d$.
13. 0; $ax^3 + bx^2 + cx + d$.
14. $7x^3 - 6x^2 + 4$; $-7x + 4x^2 - 3x^3$; $2x^2 + 7x - 4 - 4x^3$.
15. $ax^4 + bx^2 + c$; $4x^4 - 3x^2 + p$; $x^4 + qx^2 - 7$.
16. $4x^4 - 4x^3 + 4x - 1$; $-7x^4 + 4x^2 - 3x + a + 1$.

Consider examples 9 and 13 in Exercise 7.3. We observe that the polynomial 0 plays the same role as the role played by the number 0 in the set of real numbers. We know that the number 0 is the additive identity in the set of real numbers. Therefore we shall say that the polynomial 0 is the additive identity in the set of all polynomials. It is also appropriate to designate the polynomial 0 as the 'zero polynomial'. Since monomials are polynomials, the zero monomial is also the zero polynomial. The zero monomial has no degree, therefore the zero polynomial also has no degree.

Changing the signs of all the terms of the polynomial $2x^2 + 5x - 7$, we get the polynomial $-2x^2 - 5x + 7$. Adding these polynomials, we have

$$\begin{array}{r}
 + \quad 2x^2 + 5x - 7 \\
 -2x^2 - 5x + 7 \\
 \hline
 0
 \end{array}$$

The sum is the zero polynomial. Hence $-2x^2 - 5x + 7$ can be looked upon as the additive inverse of the polynomial $2x^2 + 5x - 7$. We denote the additive inverse of the polynomial $2x^2 + 5x - 7$ as $-(2x^2 + 5x - 7)$.

$$\therefore -(2x^2 + 5x - 7) = -2x^2 - 5x + 7.$$

From the above discussion, we can now say that by changing the sign of each term of a polynomial, we can obtain its additive inverse. The sum of a polynomial and its additive inverse is the zero polynomial. See examples 10, 11, 12 in Exercise 7.3.

In the set of all polynomials, the operation of addition obeys the laws of commutativity and associativity.

As in the case of numbers, subtracting one polynomial from the other means adding the additive inverse of the first to the second.

Subtraction of polynomials can now be performed by subtracting their like terms. The following worked out examples will make the procedure clear.

Ex. 7: Subtract $x^3 - 3x^2 - 2x + 6$ from $3x^3 - 5x^2 + 7x + 3$.

Subtracting the polynomial $x^3 - 3x^2 - 2x + 6$ means adding its additive inverse. The additive inverse of this polynomial can be obtained by changing the sign of each term of this polynomial.

$\therefore -x^3 + 3x^2 + 2x - 6$ is the required additive inverse.

$$\begin{aligned} &\therefore (3x^3 - 5x^2 + 7x + 3) - (x^3 - 3x^2 - 2x + 6) \\ &= (3x^3 - 5x^2 + 7x + 3) + (-x^3 + 3x^2 + 2x - 6) \\ &= (3x^3 - x^3) + (-5x^2 + 3x^2) + (7x + 2x) + (3 - 6) \\ &= 2x^3 - 2x^2 + 9x - 3 \end{aligned}$$

As in the case of addition many times it is convenient, to perform subtraction, by writing the polynomials, one below the other.

Ex. 8: Subtract $3x^4 + 3 - 4x + 5x^3$ from $5x^4 - 3x^2 + 7 + 2x^3$.

Write the first polynomial below the second, in such a way that the like terms in these polynomials appear one below the other. Change the signs of all the terms in the polynomial written below.

$$\begin{array}{r} 5x^4 + 2x^3 - 3x^2 + 0x + 7 \\ - \quad 3x^4 + 5x^3 + 0x^2 - 4x + 3 \\ \hline 2x^4 - 3x^3 - 3x^2 + 4x + 4 \end{array}$$

Subtraction can also be performed by writing the polynomials in the coefficient form.

Ex. 9: Subtract $7x + 5x^3 - 3 - 2x^4$ from $7x^4 + 2x^2 + 5 - 6x^3$.

Convert the polynomials in the coefficient form. Then write the first polynomial below the second in such a way that the coefficients of like terms appear one below the other. The first polynomial is to be subtracted from the second, hence change the sign of each lower coefficient.

$$\begin{array}{rrrrr} & 7 & -6 & 2 & 0 & 5 \\ - & -2 & 5 & 0 & 7 & -3 \\ + & & - & & - & + \\ \hline & 9 & -11 & 2 & -7 & 8 \end{array}$$

The number of coefficients in this remainder is 5.

$\therefore$ the degree of the remainder is 4.

$\therefore$ remainder $= 9x^4 - 11x^3 + 2x^2 - 7x + 8$.

Ex. 10: Subtract $5x^4 + qx^2 + 5x - r$ from $ax^4 + 7x^3 - 12x + p$.

$$\begin{array}{rrrrr} & a & 7 & 0 & -12 & p \\ - & 5 & 0 & q & 5 & -r \\ + & & - & - & - & + \\ \hline (a-5) & 7 & -q & -17 & (p+r) \end{array}$$

The number of coefficients in this remainder is 5.

$\therefore$ the degree of the remainder is 4.

$\therefore$ remainder $= (a-5)x^4 + 7x^3 - qx^2 - 17x + (p+r)$.

EXERCISE 7.4

- Subtract the second polynomial from the first.
 - $4x^2 - 7x + 1$; $x^2 - 2x + 1$
 - $3x^4 + 6x^3 - 7x^2 + 3x - 4$; $4 - 3x^4$
 - $7x^3 - 5x^2 + 3x - 7$; $4 - 5x - 6x^2 + 3x^3$
 - $12x - 3x^2 + 4x^3$; $5 - 3x + 7x^2 + 4x^3$
 - $-3x^3 - 2x^2 + x - 4$; $3x^3 + 2x^2 - x + 4$
 - $5x^6 - 3x^3 + x - 7$; $3x^5 + 2x^4 - x^2$
 - $6x^4 + 7x^2 - px + q$; $3x^3 - 2x^2 + qx + p$
 - $ax^2 + bx + c$; $px^2 + qx + r$
- Simplify.
 - $(4x^2 - 5x + 1) + (7x^2 + 2x - 3) - (5x^2 + 3x - 2)$
 - $(7x^3 + 5x^2 - 6x + 9) - (4x^3 + \frac{1}{2}x^2 + 5x - 1) + (\frac{3}{2}x^2 - 3x^3 + x - 7)$
 - $(\frac{5}{3}x^4 - 2x^2 + 3x - 11) + (7x - 2x^3 + \frac{1}{3}x^4 + 9) + (\frac{9}{4} - 3x^2 + 5x - x^3)$
 - $(4x^4 - 28x^3 + 48x^2) - (18x^3 - 32x^2 + 17) - (6x^4 + 19x^3 - 25x + 10)$
 - $-(9x^3 + 7x^2 - 10x + 15) - (5x + 10 - 3x^2 - 4x^3) + (10x^2 - 5 + 3x^3)$
- Find the sum when $5 - x^2$ is added to the sum of the polynomials $x^3 - 3x^2 + 3x - 1$ and $3x^2 - 3x + 1$.
- Find the polynomial which when added to $3x^4 + 5x^3 - 7x + 8$, will give the sum $2x^4 + 7x^3 - 5x + 6$.
- A certain polynomial is subtracted from $6x^3 + 12x^2 - 7x$. If the result is $8x^2 + 4x - 7$, find the polynomial.
- A certain polynomial is subtracted from the additive inverse of $x^4 + 3x^3 + 7x - 8$. If the remainder is $2x^3 + 5x^2 - 3$, find the polynomial.
- Subtract $5x^3 + 3x^2 - 2x + 1$ from $7x^3 + 6x^2 + 5x + 3$ and add $3x^3 - 4x^2 + 2x - 3$ to the remainder.
- Find the polynomial which when added to the sum of $2x^3 - 3x^2 - 5x + 7$ and $x^3 + 3x^2 + 3x + 1$, will give the sum $3x^3 - 8x^2 + 5$.
- A certain polynomial is added to the sum of $7 - 3x^2 - 4x + 10x^3$ and $8x + 2x^3 - 2x^2 + 3$. If the resulting sum is $14x^3 - x^2 + 2x - 7$, find the polynomial.
- $u = x^2 + 2x + 1$; $v = x^2 - 2x + 1$ and $w = 2x^2 + 2$. Express $u + v - w$ as a polynomial in x .
- If the sum of $5x^4 + 3x^2 - 7x + 8$ and $ax^4 + bx^3 + cx^2 + dx + e$ is $12x^4 + 7x^3 - 6x^2 + 9x - 11$, find a, b, c, d, e .
- If the remainder, when a certain polynomial is subtracted from $x^3 + 5x - 6$ is zero, find the polynomial.
- If the lengths of the sides of a triangle are $a - 2$; $2a + 6$; $3a - 2$, find its perimeter.
- I had Rs. $m^3 + 2m^2 + 2m$. I withdrew Rs. $3m^3 + 4m^2 - 6m$ from the bank and spent Rs. $2m^3 - 6m^2 + 3m$. How much money is left with me?

15. Suresh purchased note-books worth Rs. $5a^2 - 4a + 2$; pencils worth Rs. $\frac{2}{3}a^2 + \frac{5}{2}a - 3$ and books worth Rs. $8a^2 + 7a$. If he had Rs. $14a^2 + 6a + 7$, how much money is now left with him?
16. What is the additive inverse of (i) the zero polynomial and (ii) the constant polynomial c ?
17. What is the additive inverse of a polynomial of degree 0?
18. State the relation between the degree of a polynomial and the number of its terms.
19. What is the relation between the degrees of a polynomial and its additive inverse?
20. If the sum of two polynomials is the zero polynomial, state the relation between them.

§ 7.4 Multiplication of Polynomials :

We have assumed that the addition and multiplication of the symbols x, y etc. used in polynomials and the real numbers obey the laws of commutativity, associativity and distributivity.

Now let us multiply $5x^3$ and $2x^2$.

$$\begin{aligned}
 (5x^3) \times (2x^2) &= [5 \times (x^3 \times 2)] \times x^2 \quad \dots \text{(Associativity of Multiplication)} \\
 &= [5 \times (2 \times x^3)] \times x^2 \quad \dots \text{(Commutativity of Multiplication)} \\
 &= (5 \times 2) \times (x^3 \times x^2) \quad \dots \text{(Associativity of Multiplication)} \\
 &= 10 \times x^{3+2} \quad \dots (x^m \times x^n = x^{m+n}) \\
 &= 10x^5
 \end{aligned}$$

In general we can say that $ax^m \times bx^n = abx^{m+n}$.

The above law and distributivity law are mainly used while multiplying two polynomials. The like terms are then taken together and added by making use of commutativity and associativity of addition. The method for obtaining the product of two polynomials will now be clear to you from the following solved examples.

Ex. 1 : Multiply $x^2 + 2x - 4$ and $3x + 5$.

$$\begin{aligned}
 &(x^2 + 2x - 4)(3x + 5) \\
 &= (x^2 + 2x - 4)(3x) + (x^2 + 2x - 4)(5) \quad \dots \text{(Distributivity)} \\
 &= (3x^3 + 6x^2 - 12x) + (5x^2 + 10x - 20) \quad \dots \text{(Distributivity)} \\
 &= (3x^3) + (6x^2 + 5x^2) + (-12x + 10x) + (-20) \\
 &= 3x^3 + 11x^2 - 2x - 20.
 \end{aligned}$$

The following is a more convenient method of multiplication.

$$\begin{array}{r}
 x^2 + 2x - 4 \\
 \underline{3x + 5} \\
 3x^3 + 6x^2 - 12x \quad \dots \text{(Multiplying by } 3x) \\
 + 5x^2 + 10x - 20 \quad \dots \text{(Multiplying by } 5) \\
 \hline
 3x^3 + 11x^2 - 2x - 20 \quad \dots \text{(Adding the like terms)}
 \end{array}$$

Note that we have made use of the same laws as above in this method also.

Ex. 2: Multiply $3x^3 - 2x + 1$ and $5x^2 - 3x - 2$.

$$\begin{array}{r} 3x^3 + 0x^2 - 2x + 1 \\ 5x^2 - 3x - 2 \end{array}$$

$$\begin{array}{r} 15x^5 + 0x^4 - 10x^3 + 5x^2 \\ - 9x^4 + 0x^3 + 6x^2 - 3x \\ - 6x^3 + 0x^2 + 4x - 2 \end{array}$$

$$15x^5 - 9x^4 - 16x^3 + 11x^2 + x - 2$$

.. (Multiplying by $5x^2$)

.. (Multiplying by $-3x$)

.. (Multiplying by -2)

.. (Adding like terms)

In Ex. 1 the degree of the first polynomial is 2 and the degree of the second polynomial is 1, while the degree of their product is $(2 + 1) = 3$. In Ex. 2 the degrees of the two given polynomials are 3 and 2, while the degree of their product is $(3 + 2) = 5$.

We can generalise this observation as—

The product of two polynomials of degrees m and n respectively is a polynomial of degree $m + n$.

EXERCISE 7.5

Multiply the polynomials in examples 1 to 16 and state the degrees of their products.

1. $(3x^2 + 2x + 5); (x + 4)$
2. $(6x^3 + x + 4); (x - 2)$
3. $(-2x^2 + 3); (3x^3 - 2x + 4)$
4. $(4x^2 + 2x + 3); (x^2 + 3x + 1)$
5. $(-7x^2 - 3x + 2); (5x^2 - 6x - 2)$
6. $(x - 2); (x^2 + 2x + 4)$
7. $(x - a); (x^3 + ax^2 + a^2x + a^3)$
8. $(x + p); (x^4 - px^3 + p^2x^2 - p^3x + p^4)$
9. $(2x - 3); (4x^2 + 6x + 9)$
10. $(ax^7 + 6x^5 + bx^3 - 4x^2 + 5); (4x^3 + px - 3)$
11. $(px^6 + qx^2 - 7x + r); (ax^4 + bx^3 + d)$
12. $(mx^3 + nx^2 + px); (7x^2 - 5x + a + 1)$
13. $(5x^2 + 4); (25x^4 + 20x^2 + 16)$
14. $(2x^2 - 12x + 7); (6x^3 + 7x^2 - 4x + 3)$
15. $(3x^4 - 8x^3 - 5x^2 + 7x - 1); (x^2 - 3x + 4)$
16. $(5x^6 - 3x^5 + 8x - 9); (ax^3 - 4x + p)$
17. m and n are the degrees of two polynomials. What is the degree of their product?
18. Multiply $x^3 + 2x^2 - 1$ by the constant polynomial 1. Is the product the same polynomial? How will you designate the polynomial 1 w.r.t. multiplication from this example?
19. Multiply $x^4 + 3x^3 - 2x + 4$ by the zero polynomial (0). State the property suggested from this example.
20. Show that $(ax^3 + bx^2 + cx + d)(px^2 + qx + r) = apx^5 + (bp + aq)x^4 + (cp + bq + ar)x^3 + (dp + cq + br)x^2 + (dq + cr)x + dr$.

§ 7.5 Division of Polynomials :

We have seen how to add, subtract and multiply two polynomials. Now we shall see how to divide one polynomial by another. Recollect the process of division of one integer by another. The procedure adopted for division in the set of polynomials, is quite similar to that in the set of integers.

Ex. 1: Divide $3x^4 - 6x^2 + 9x - 1$ by $3x^2$.

$$\begin{array}{r}
 x^2 - 2 \\
 3x^2 \overline{) 3x^4 - 6x^2 + 9x - 1} \\
 \underline{- 3x^4} \\
 - 6x^2 + 9x - 1 \\
 \underline{- 6x^2} \\
 + 9x - 1
 \end{array}$$

..(Product of x^2 and $3x^2$)
 ..(First remainder)
 ..(Product of -2 and $3x^2$)
 ..(Second remainder)

Now the remainder $9x - 1$ is a polynomial of degree 1 which is less than degree 2 of the divisor $3x^2$. Therefore the process ends at this stage.

$3x^4 - 6x^2 + 9x - 1$ is called **dividend**; $3x^2$ is called **divisor**; $x^2 - 2$ is called **quotient**; $9x - 1$ is called **remainder**.

We can write the relation between these as

$$3x^4 - 6x^2 + 9x - 1 = (3x^2) \times (x^2 - 2) + (9x - 1).$$

Ex. 2: Divide $12x^3 + 6x^2 + 3x + 3$ by $3x + 3$.

$$\begin{array}{r}
 4x^2 - 2x + 3 \\
 3x+3 \overline{) 12x^3 + 6x^2 + 3x + 3} \\
 \underline{12x^3 + 12x^2} \\
 - 6x^2 + 3x + 3 \\
 \underline{- 6x^2 - 6x} \\
 + 9x + 3 \\
 \underline{9x + 9} \\
 - 6
 \end{array}$$

..(Product of $4x^2$ and $3x + 3$)
 ..(First remainder)
 ..(Product of $-2x$ and $3x + 3$)
 ..(Second remainder)
 ..(Product of 3 and $3x + 3$)
 ..(Third remainder)

The third remainder -6 is a polynomial of degree 0 which is less than the degree 1 of the divisor $3x + 3$. Therefore, the process ends at this stage.

Thus, if $12x^3 + 6x^2 + 3x + 3$ is divided by $3x + 3$, the quotient is $4x^2 - 2x + 3$ and the remainder is -6 and we can express the dividend as—

$$12x^3 + 6x^2 + 3x + 3 = (3x + 3) \times (4x^2 - 2x + 3) + (-6).$$

Ex. 3: Divide $6x^3 - 5x^2 + 2x - 8$ by $2x^2 + x + 2$.

$$\begin{array}{r}
 3x - 4 \\
 2x^2 + x + 2 \overline{) 6x^3 - 5x^2 + 2x - 8} \\
 \underline{6x^3 + 3x^2 + 6x} \\
 - 8x^2 - 4x - 8 \\
 \underline{- 8x^2 - 4x - 8} \\
 + + \\
 0
 \end{array}$$

..(Product of $3x$ and $2x^2 + x + 2$)
 ..(First remainder)
 ..(Product of -4 and $2x^2 + x + 2$)
 ..(Second remainder)

Thus, if $6x^3 - 5x^2 + 2x - 8$ is divided by $2x^2 + x + 2$, the quotient is $3x - 4$ and the remainder is 0. We can express the dividend as—

$$6x^3 - 5x^2 + 2x - 8 = (2x^2 + x + 2) \times (3x - 4) + 0.$$

Ex. 4: Divide $4x^4 - 3x^2 + x + 2$ by $2x^2 + x + 1$.

$$\begin{array}{r}
 2x^2 - x - 2 \\
 2x^2 + x + 1 \overline{) 4x^4 + 0x^3 - 3x^2 + x + 2} \\
 \underline{4x^4 + 2x^3 + 2x^2} \quad \dots \text{(Product of } 2x^2 \text{ and } 2x^2 + x + 1) \\
 -2x^3 - 5x^2 + x + 2 \quad \dots \text{(First remainder)} \\
 \underline{-2x^3 - x^2 - x} \quad \dots \text{(Product of } -x \text{ and } 2x^2 + x + 1) \\
 +x^2 + 2x + 2 \quad \dots \text{(Second remainder)} \\
 \underline{-4x^2 - 2x - 2} \quad \dots \text{(Product of } -2 \text{ and } 2x^2 + x + 1) \\
 4x + 4 \quad \dots \text{(Third remainder)}
 \end{array}$$

Thus, if $4x^4 - 3x^2 + x + 2$ is divided by $2x^2 + x + 1$, the quotient is $2x^2 - x - 2$ and the remainder is $4x + 4$. We can express the dividend as—

$$4x^4 - 3x^2 + x + 2 = (2x^2 + x + 1) \times (2x^2 - x - 2) + (4x + 4).$$

From the above examples, we observe that:

1. We divide one polynomial by another, if the degree of the former is greater than or equal to the degree of the latter.
2. The degree of the quotient is equal to, the degree of the dividend minus the degree of the divisor.
3. The remainder is either the zero polynomial or a polynomial of degree less than that of the divisor.

EXERCISE 7-6

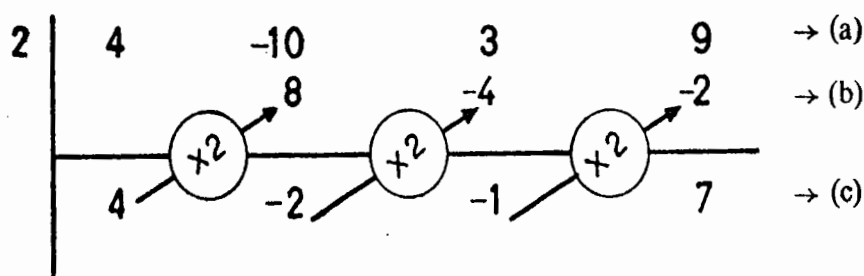
Perform the following divisions.

1. $(8x^2 + 4x) \div (2x)$
2. $(18x^5 + 9x^4 - 36x^3) \div (9x^2)$
3. $(x^2 - 4) \div (x - 2)$
4. $(x^2 - x - 20) \div (x - 5)$
5. $(-15x^2 + 5x) \div (-3x + 1)$
6. $(48x^3 - 6x^2 + 3) \div (4x^2)$
7. $(x^4 + x^2 + 1) \div (x^2 + x + 1)$
8. $(3x^3 + 4x^2 - 16x + 2) \div (x - 2)$
9. $(25x^2 + 16) \div (5x + 4)$
10. $(x^3 - 2x^2 - 1) \div (x - 1)$
11. $(4y^3 + 3y + 7) \div (3y^2 - 3y + 5)$
12. $(6x^3 - x^2 - 22x + 15) \div (3x^2 + 4x - 5)$
13. $(6y^4 - 2y^2 + 5y - 7) \div (3y^2 + 4y - 5)$
14. $(12x^3 - 11x^2 + 9) \div (4x - 1)$
15. $(3x^4 - 3x + 7) \div (x + 2)$
16. $(3x^5 - 2x^4 + 3x^2 - x + 5) \div (x^3 - 2x^2 + 3)$
17. $(x^3 - 8) \div (x - 2)$
18. $(x^4 - a^4) \div (x - a)$
19. $(x^3 + 3ax^2 + 3a^2x + a^3) \div (x + a)$
20. $(a^2x^2 + 2abx + c) \div (ax + b)$
21. $(ax^2 + 2bx + c) \div (x + 3)$
22. $(x^6 - 64a^6) \div (x - 2a)$
23. $(x - 10 + 3x^2) \div (x + 2)$
24. $(ax^2 + bx + c) \div (x - p)$

§ 7-6 Synthetic Division :

You can now divide one polynomial by another when they are expressed in the index form. You also know how to convert a polynomial, given in the index form into its coefficient form. Now we shall discuss an easier method of division when the divisor is $x \pm a$. This method of division is called the method of **synthetic division**. The example on the next page will illustrate this method.

Ex. 1: Divide $4x^3 - 10x^2 + 3x + 9$ by $x - 2$.



Explanation :

1. Write the polynomial in the dividend in the coefficient form, in row (a).
2. Divisor is $x - 2$. $\therefore$ In the place of the divisor write 2, the additive inverse of -2 .
3. Draw a horizontal line below the dividend, after leaving some space for the row (b). Below this line write the first coefficient 4 of the dividend in the row (c).
4. Multiply the first coefficient 4 in the row (c) by 2, the number in the place of the divisor and write the product 8, below the second coefficient in row (b). Add the second coefficients and write their sum -2 , in the row (c) as the second coefficient.
5. Multiply the second coefficient -2 in the row (c) again by 2, the number in the place of the divisor and write the product -4 , below the third coefficient in row (b). Add the third coefficients and write their sum -1 in the row (c) as the third coefficient.
6. Multiply the third coefficient -1 in the row (c) again by 2, the number in the place of the divisor and write the product -2 , below the fourth coefficient in row (b). Add the fourth coefficients and write their sum 7 in the row (c) as the fourth coefficient.
7. This last coefficient 7 in the row (c) is the remainder and the rest of the coefficients in the order give the quotient in the coefficient form.

$$\therefore \text{quotient} = (4, -2, -1) = 4x^2 - 2x - 1$$

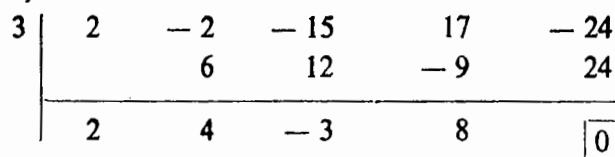
$$\text{remainder} = 7$$

We can then write the dividend as—

$$4x^3 - 10x^2 + 3x + 9 = (x - 2) \times (4x^2 - 2x - 1) + 7.$$

Ex. 2: Divide $2x^4 - 2x^3 - 15x^2 + 17x - 24$ by $x - 3$.

The divisor is $x - 3$ which contains -3 as the constant term, $\therefore$ in the place of the divisor we take 3, the additive inverse of -3 .



$$\therefore \text{remainder} = 0$$

$$\text{and quotient} = (2, 4, -3, 8) = 2x^3 + 4x^2 - 3x + 8$$

The dividend can now be expressed as—

$$2x^4 - 2x^3 - 15x^2 + 17x - 24 = (x - 3)(2x^3 + 4x^2 - 3x + 8) + 0.$$

Ex. 3: Divide $4x^5 + 7x^4 - 5x^3 + 2x^2 + 15x + 3$ by $x + 2$.

The divisor is $x + 2$ which contains 2 as the constant term, $\therefore$ in the place of the divisor we take -2 , the additive inverse of 2.

$$\begin{array}{r|rrrrrrr} -2 & 4 & 7 & -5 & 2 & 15 & 3 \\ & & -8 & 2 & 6 & -16 & 2 \\ \hline & 4 & -1 & -3 & 8 & -1 & \boxed{5} \end{array}$$

$\therefore$ remainder = 5

and quotient $= (4, -1, -3, 8, -1) = 4x^4 - x^3 - 3x^2 + 8x - 1$

Ex. 4: Divide $5x^5 + 13x^4 - 4x^3 - 18x + 6$ by $x + 3$.

$$\begin{array}{r|rrrrrrr} -3 & 5 & 13 & -4 & 0 & -18 & 6 \\ & & -15 & 6 & -6 & 18 & 0 \\ \hline & 5 & -2 & 2 & -6 & 0 & \boxed{6} \end{array}$$

$\therefore$ remainder = 6

and quotient $= (5, -2, 2, -6, 0) = 5x^4 - 2x^3 + 2x^2 - 6x$.

EXERCISE 7.7

Use synthetic division to find the following quotients.

1. $(x^3 - 5x^2 + 4x + 7) \div (x - 1)$
2. $(3x^3 + 10x^2 + 9x + 2) \div (x - 2)$
3. $(5x^3 - 12x^2 - 16) \div (x + 2)$
4. $(2x^3 - 4x + 3) \div (x + 3)$
5. $(3x^4 - 8x^3 - 5x^2 + 7x - 1) \div (x - 4)$
6. $(3x^4 + 8x^3 + 9x^2 + 7x - 2) \div (x + 4)$
7. $(x^4 - 7x^2 + 2x - 10) \div (x + 3)$
8. $(x^5 - 17x^3 + 37x^2 - 11x + 7) \div (x + 5)$
9. $(x^4 - 81) \div (x - 3)$
10. $(x^4 + 64) \div (x + 4)$
11. $(x^5 - 32) \div (x - 2)$
12. $(3x^3 + 4x^2 - 16x + 2) \div (x - 2)$
13. $(x^3 - 2x^2 - 1) \div (x + 1)$
14. $(3x^4 - 3x + 7) \div (x + 2)$
15. $(x - 10 + 3x^2) \div (x + 2)$
16. $(x^3 + 3ax^2 + 3a^2x + a^3) \div (x + a)$
17. $(ax^2 + 2bx + c) \div (x + 3)$
18. $(ax^2 + bx + c) \div (x - p)$
19. $(3x^4 - 2x^3 - 6x^2 + 19x - 10) \div (x - 2)$
20. $(15x^4 - 6x^3 - 5x^2 - 12) \div (x - 3)$
21. $(7x^4 - 5x^3 + 35x + 15) \div (x - 5)$
22. $(4x^5 - 3x^3 + x^2 - 5) \div (x - 2)$
23. $(25x^2 + 16) \div (x + 4)$
24. $(12x^3 - 11x^2 + 9) \div (x - 1)$

§ 7.7 Division Algorithm :

Recall division algorithm for integers.

If n is a non-negative integer and a is a positive integer, then there exist integers q and r such that

$$n = a \times q + r, \text{ where } 0 \leq r < a$$

This can also be written as

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

where $0 \leq \text{Remainder} < \text{Divisor}$.

Division algorithm for polynomials is similar, while solving examples from exercise 7.6 and 7.7 you might have observed the following—

If dividend and divisor are polynomials, then quotient and remainder are also polynomials and the following relation holds amongst them—

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

where remainder is either the Zero polynomial or
 $0 \leq \text{degree of remainder} < \text{degree of divisor}$.

This is the division algorithm for polynomials.

For convenience, let us denote the polynomials by capital letters A, B, C, then the division algorithm for polynomials can be stated as follows—

If P is a polynomial and A is a polynomial of degree ≥ 1 , then there are polynomials Q and R such that

$$P = A \times Q + R$$

where R is either the zero polynomial or degree of R $<$ degree of A.

Ex. 1: Divide $4x^4 - 3x^2 + x + 2$ by $2x^2 + x + 1$ and express the dividend in the form

$$P = A \times Q + R.$$

After actual division, we get

$$\text{Dividend} = P = 4x^4 - 3x^2 + x + 2$$

$$\text{Divisor} = A = 2x^2 + x + 1$$

$$\text{Quotient} = Q = 2x^2 - x - 2$$

$$\text{and Remainder} = R = 4x + 4$$

$\therefore$ By division algorithm,

$$4x^4 - 3x^2 + x + 2 = (2x^2 + x + 1) \times (2x^2 - x - 2) + (4x + 4).$$

Ex. 2: Divide $4x^3 - 10x^2 + 3x + 9$ by $x - 2$ and express the dividend in the form

$$P = A \times Q + R$$

2	4	-10	3	9
		8	-4	-2
	4	-2	-1	7

$$\therefore \text{Dividend} = P = 4x^3 - 10x^2 + 3x + 9$$

$$\text{Divisor} = A = x - 2$$

$$\text{Quotient} = Q = 4x^2 - 2x - 1$$

$$\text{Remainder} = R = 7$$

$\therefore$ By division algorithm,

$$4x^3 - 10x^2 + 3x + 9 = (x - 2) \times (4x^2 - 2x - 1) + 7.$$

Express the dividends in the examples from Exercises 7.6 and 7.7 in the form

$$P = A \times Q + R$$

§ 7.8 Divisibility in Polynomials :

There is a close similarity between the division in integers and division in polynomials. We can extend this similarity to the concept of 'divisibility'.

An integer n is said to be divisible by a non-zero integer a , if there exists an integer q such that $n = a \times q$.

By division algorithm, if n and a ($\neq 0$) are given integers, then we can express n as

$$n = a \times q + r.$$

If $r = 0$, then it is clear that n is divisible by a , and ' n is divisible by a ' can be expressed by saying that 'when n is divided by a , the remainder is 0'.

In the same way, we can consider divisibility in polynomials as follows—

By division algorithm for polynomials (§ 7.7)

$$P = A \times Q + R$$

where (i) R is either the zero polynomial or (ii) degree of $R \leq$ degree of A .

If R is the zero polynomial (0), then $P = A \times Q + 0$ i.e. $P = A \times Q$.

In this case, P is said to be divisible by A . Thus if $R = 0$, then P is divisible by A and if $R \neq 0$, then P is not divisible by A .

Ex. 1: Is $4x^4 - 3x^3 - 7x^2 + 5x + 3$ divisible by $4x^2 + x - 2$?

After actual division, we find that remainder = $R = 4x + 1 \neq 0$.

$\therefore 4x^4 - 3x^3 - 7x^2 + 5x + 3$ is not divisible by $4x^2 + x - 2$.

Ex. 2: Is $2x^4 - 2x^3 - 15x^2 + 17x - 24$ divisible by $x - 3$?

3	2	-2	-15	17	-24
		6	12	-9	24
	2	4	-3	8	0

Here the remainder = 0 $\therefore 2x^4 - 2x^3 - 15x^2 + 17x - 24$ is divisible by $x - 3$.

EXERCISE 7.8

By synthetic division decide whether the first polynomial is divisible by the second in each of the following examples.

- | | |
|---|-------------------------------------|
| 1. $x^3 - x - 12$; $x + 3$ | 2. $6x^3 + 13x^2 - 4$; $x + 2$ |
| 3. $3x^5 - 2x^3 + 8x + 9$; $x + 1$ | 4. $2x^5 - 3x^2 + x - 1$; $x - 2$ |
| 5. $3x^3 - 5x^2 + x - 7$; $x + 2$ | 6. $x^4 - 3x^2 + 2x + 2$; $x + 2$ |
| 7. $x^3 - 2x^2 + 5x - 10$; $x - 2$ | 8. $3x^3 + x^2 - 5x + 6$; $x + 2$ |
| 9. $x^3 - 2x^2 - x + 28$; $x - 4$ | 10. $x^3 + 3x^2 - 6x - 2$; $x + 2$ |
| 11. $x^3 + 3ax^2 + 3a^2x + a^3$; $x + a$ | 12. $x^5 - 32$; $x - 2$ |
| 13. $x^5 + 32$; $x + 2$ | 14. $x^4 - 81$; $x - 3$ |
| | 15. $x^4 - 81$; $x + 3$ |

§ 7.9 Value of a Polynomial :

Consider the polynomial $P = 2x^2 - 5x + 7$ in x . Let us replace x successively by the numbers 2, 3, -4, -5, a , $-a$, and write the polynomial in powers of these numbers. Let us denote the polynomial written in powers of 2 by $P(2)$.

Then $P(2) = 2(2)^2 - 5(2) + 7$	$P(3) = 2(3)^2 - 5(3) + 7$
$P(-4) = 2(-4)^2 - 5(-4) + 7$	$P(-5) = 2(-5)^2 - 5(-5) + 7$
$P(a) = 2(a)^2 - 5(a) + 7$	$P(-a) = 2(-a)^2 - 5(-a) + 7$

$P(2)$, $P(3)$, $P(-4)$ etc. are called the values of the polynomial P , for $x = 2, 3, -4$ etc. respectively.

Definition : If P is a polynomial in x and x is replaced by a certain number a , then the number represented by the symbol $P(a)$ is called the **Value** of the polynomial P when $x = a$.

Find the values of the polynomial P given above when $x = 2$; $x = 3$; $x = -4$; $x = -5$; $x = a$ and $x = -a$.

Ex. 1: Find the value of the polynomial $x^3 - 3x^2 - 10x + 24$, when $x = -3$.

$$\text{Let } A = x^3 - 3x^2 - 10x + 24$$

$$\therefore A(-3) = (-3)^3 - 3(-3)^2 - 10(-3) + 24 \\ = -27 - 27 + 30 + 24 = -54 + 54 = 0$$

$$\therefore A(-3) = 0.$$

Ex. 2: Find the value of the polynomial $-5x^3 + \frac{7}{2}x^2 + \frac{6}{5}x - 3$ when $x = a$.

$$\text{Let } B = -5x^3 + \frac{7}{2}x^2 + \frac{6}{5}x - 3$$

$$\therefore B(a) = -5(a)^3 + \frac{7}{2}(a)^2 + \frac{6}{5}(a) - 3 = -5a^3 + \frac{7}{2}a^2 + \frac{6}{5}a - 3$$

Note here that a is a given real number and $B(a)$ is the value of the polynomial B , when x is replaced by the real number a . Coefficients a, b, c, d, p, q, r etc., in polynomials represent real numbers.

EXERCISE 7.9

- Find the values of the polynomial $x^2 + 3x - 2$ when (i) $x = 2$
(ii) $x = -2$ (iii) $x = 0$ (iv) $x = a$ (v) $x = 2b$ (vi) $x = a + 2b$.
- Find the values of the polynomial $-2x^2 - \frac{1}{2}x + 3$ when (i) $x = 1$
(ii) $x = -1$ (iii) $x = 0$ (iv) $x = -3$ (v) $x = a$ (vi) $x = \frac{1}{2}b$.
- Find the values of the polynomial $-4y^3 + y$ when (i) $y = 5$ (ii) $y = 0$
(iii) $y = -1$ (iv) $y = c$ (v) $y = 3c$ (vi) $y = a + \frac{1}{a}$ ($a \neq 0$).
- Find the values of the polynomial $3y^3 + 2y^2 + 5$ when (i) $y = -1$
(ii) $y = -3$ (iii) $y = 0$ (iv) $y = d$ (v) $y = a - \frac{1}{a}$ ($a \neq 0$).
- Find the values of the polynomial $x^3 - 2ax^2 + 4a^2x - 8a^3$ when (i) $x = a$
(ii) $x = 3a$.
- Find the values of the polynomial $x^2 + 2ax - 2a - 2x + a^2 + 1$ when
(i) $x = 1$ (ii) $x = 0$ (iii) $x = -a$.
- Write a polynomial of degree 5, whose value when $x = 0$ is -7 .
- Write two polynomials of first degree such that the value of each when $x = 2$ is 3.
- If the value of the polynomial $ax^2 + 6x + 5$ when $x = 3$ is 41, find a .
- If the value of the polynomial $3x^2 + bx - 2$ when $x = -2$ is 0, find b .
- If the value of the polynomial $ax^3 + x^2 + bx + 8$ when $x = 3$ is 41 and when $x = -4$ is 0, find a and b .
- If the value of the polynomial $2x^3 + px^2 - 10x + q$ when $x = 1$ is 9 and when $x = -2$ is 0, find p and q .

§ 7.10 Remainder Theorem :

Let us divide $3x^3 - 2x^2 + x - 9$ by $x - 2$.

$$\begin{array}{r|rrrr} 2 & 3 & -2 & 1 & -9 \\ & & 6 & 8 & 18 \\ \hline & 3 & 4 & 9 & \boxed{9} \end{array}$$

Here quotient $= 3x^2 + 4x + 9$

remainder $= 9$.

Now let us find the value of this polynomial when $x = 2$.

$$\text{Let } A = 3x^3 - 2x^2 + x - 9$$

$$\begin{aligned} \therefore A(2) &= 3(2)^3 - 2(2)^2 + (2) - 9 \\ &= 3 \times 8 - 2 \times 4 + 2 - 9 \\ &= 24 - 8 + 2 - 9 \\ &= 9 \end{aligned}$$

This value is the same as the above remainder. Therefore we feel that there must be some relation between them. Let us try to find it.

By division algorithm,

$3x^3 - 2x^2 + x - 9 = (x - 2)(3x^2 + 4x + 9) + 9$, where $3x^2 + 4x + 9$ is the quotient and 9 is the remainder. In this equality, replace x by 2. Then the l. h. s. is the value of the polynomial A when $x = 2$. Thus the value of A when $x = 2$ is the same as the remainder when A is divided by $x - 2$.

If we divide the polynomial $P = 4x^5 - 23x^3 - 17x + 1$ by $x - 3$, then the remainder is either the zero polynomial or it is a polynomial of degree less than the degree of $x - 3$ i.e. the remainder is either the zero polynomial or a polynomial of degree 0. Thus the remainder is some fixed number.

By division algorithm,

$$P = 4x^5 - 23x^3 - 17x + 1 = (x - 3) \times Q + R.$$

Replacing x by 3 in both sides, we get

$$\begin{aligned} P(3) &= (3 - 3) \times Q(3) + R \\ &= 0 \times Q(3) + R \\ &= R \end{aligned}$$

Thus the value of P when $x = 3$ is the remainder when P is divided by $x - 3$.

$$\begin{aligned} \therefore R &= P(3) \\ &= 4(3)^5 - 23(3)^3 - 17(3) + 1 \\ &= 972 - 621 - 51 + 1 \\ &= 973 - 672 = 301. \end{aligned}$$

EXERCISE 7.10

By the above method, find the remainder, when the first polynomial is divided by the second, in each of the following examples.

1. $3x^3 - 5x^2 + 3x - 14$; $x - 2$
2. $5x^3 - 11x^2 - 33$; $x - 3$
3. $7x^4 + 12x^3 + 6x - 4$; $x + 2$
4. $4x^4 - 39x^2 + 23$; $x + 3$
5. $x^5 - 32$; $x - 2$

Now, let P be any polynomial and $x - a$ be another. By division algorithm, there exist polynomials Q and R such that

$$P = (x - a)Q + R \quad \dots\dots\dots (1)$$

where R is zero or a non-zero number and R is the remainder when P is divided by $x - a$.

The values of the polynomials P and Q when $x = a$ are respectively $P(a)$ and $Q(a)$. In order to find $P(a)$, we have to find the values of the polynomials in the r.h.s of (1) and (1) will become,

$$\begin{aligned} P(a) &= (a - a)Q(a) + R \\ \therefore P(a) &= R \end{aligned}$$

This is called the remainder theorem.

Remainder Theorem : If a polynomial P in x is divided by $x - a$, then the remainder is the value of the polynomial P when $x = a$.

Ex. 1: Find the remainder when $x^4 - 13x^2 + 10x - 3$ is divided by $x - 2$.

Taking $x - 2 = 0$; $x = 2$

Let $P = x^4 - 13x^2 + 10x - 3$

$\therefore$ By remainder theorem,

$$\begin{aligned} \text{remainder} = P(2) &= (2)^4 - 13(2)^2 + 10(2) - 3 \\ &= 16 - 52 + 20 - 3 \\ &= 36 - 55 = -19. \end{aligned}$$

Ex. 2: Find the remainder when $2x^5 - 23x^3 + 47x + 1$ is divided by $x + 3$.

Taking $x + 3 = 0$, $x = -3$.

Let $P = 2x^5 - 23x^3 + 47x + 1$

$$\begin{aligned} \therefore \text{remainder} &= P(-3) \\ &= 2(-3)^5 - 23(-3)^3 + 47(-3) + 1 \\ &= 2(-243) - 23(-27) + 47(-3) + 1 \\ &= -486 + 621 - 141 + 1 \\ &= 622 - 627 = -5. \end{aligned}$$

By remainder theorem, calculations become lengthy. But by the synthetic division calculations reduce substantially and we can obtain the remainder and the quotient at one stroke. Let us solve the above example by the synthetic division.

$$\begin{array}{r|rrrrrr} -3 & 2 & 0 & -23 & 0 & 47 & 1 \\ & & -6 & 18 & 15 & -45 & -6 \\ \hline & 2 & -6 & -5 & 15 & 2 & -5 \end{array}$$

Ex. 3: If the polynomial $3x^4 + 3x^3 - 14x^2 - 8x + p$ is divided by $x + 2$, the remainder is 1, find p .

Let $A = 3x^4 + 3x^3 - 14x^2 - 8x + p$.

$$\begin{aligned} \therefore \text{remainder} &= A(-2) \\ &= 3(-2)^4 + 3(-2)^3 - 14(-2)^2 - 8(-2) + p \\ &= 48 - 24 - 56 + 16 + p \\ &= 64 - 80 + p = p - 16 \\ \therefore p - 16 &= 1 \quad \therefore p = 17. \end{aligned}$$

Another Method :

$$\begin{array}{r|rrrrr}
 -2 & 3 & 3 & -14 & -8 & p \\
 & & -6 & 6 & 16 & -16 \\
 \hline
 & 3 & -3 & -8 & 8 & \boxed{p-16}
 \end{array}$$

$$\therefore \text{remainder} = p - 16$$

$$\therefore p - 16 = 1$$

$$\therefore p = 17.$$

EXERCISE 7.11

By the remainder theorem, find the remainder when the first polynomial is divided by the second in examples 1 to 21. Check your answers by using synthetic division.

1. $3x^2 + 6x - 1$; $x - 2$
2. $2x^2 - 5x + 8$; $x + 3$
3. $-5x^3 + x + 9$; $x + 2$
4. $-6x^2 + 5x + 3$; $x - 3$
5. $5x^4 + 3x^3 - 7x^2 - 2x + 3$; $x - 1$
6. $5x^4 - 3x^3 - 7x^2 - x + 3$; $x + 2$
7. $x^3 - 7x^2 + 8$; $x + 1$
8. $2x^2 - 4x + 3$; $x - a$
9. $ax^3 + ax^2 - bx - b$; $x + 1$
10. $x^3 - 5x^2 - 2x + 12$; $x - 2$
11. $2x^3 - 8x^2 + 10x + 2$; $x - 3$
12. $x^3 - 60x + 180$; $x + 4$
13. $x^3 - 16x + 24$; $x - 2$
14. $4x^3 - 3x - 1$; $x + \frac{1}{2}$
15. $x^3 - 39x - 70$; $x + 2$
16. $x^3 - 3x^2 - 10x + 24$; $x - 2$
17. $x^3 + 3x^2 - 6x - 2$; $x - 2$
18. $x^4 - 2x + 5$; $x + 3$
19. $x^3 - 2x^2 + 7x - 10$; $x - 4$
20. $3x^3 + 8$; $x + 2$
21. $x^4 - 2x^2 + 5$; $x + 3$
22. When the polynomial $2x^3 - 9x^2 + 13x + p$ is divided by $x - 2$, the remainder is -4 . Find p .
23. When the polynomial $3x^4 + 5x^3 - 49x^2 + px + 40$ is divided by $x - 1$, the remainder is 10. Find p .
24. When the polynomial $2x^3 + px^2 + qx + 4$ is divided by $x - 1$, the remainder is 2 and when it is divided by $x - 3$, the remainder is 16. Find p and q .
25. When the polynomial $4x^3 - 7x^2 + ax + b$ is divided by $x - 2$, the remainder is 13 and when it is divided by $x - 1$, the remainder is 6. Find a and b .
26. Find a , if the polynomials $ax^2 - 10x + 3$ and $3x^2 + 5x + a$ are divided by $x - 4$, the remainders are equal.

§ 7.11 Factor Theorem :

Ex. 1: Let $P = 3x^3 - 2x^2 + x - 18$. Using the remainder theorem, let us find the remainder when P is divided by $x - 2$.

$$\begin{aligned}
 R &= P(2) = 3(2)^3 - 2(2)^2 + (2) - 18 \\
 &= 24 - 8 + 2 - 18 \\
 &= 26 - 26 = 0
 \end{aligned}$$

The remainder is 0. $\therefore P$ is divisible by $x - 2$. In this case, $x - 2$ is said to be a factor of P .

Ex. 2: Decide whether $x - 3$ is a factor of $P = x^4 - 13x^2 + 10x + 7$.

Here $P(3) = 1$. $\therefore P$ is not divisible by $x - 3$.

$\therefore x - 3$ is not a factor of P .

EXERCISE 7.12

Using the remainder theorem, find the remainder when the first polynomial is divided by the second in each of the following examples and state whether the second polynomial is a factor of the first.

1. $2x^3 + x^2 - 11x + 2$; $x - 2$
2. $4x^3 - 3x^2 + 2x - 2$; $x - 1$
3. $x^3 - x^2 + 14x + 6$; $x + 3$
4. $x^3 - 2x^2 - 7x + 2$; $x + 2$
5. $x^3 - x^2 - 13x + 4$; $x - 4$
6. $x^3 - 60x + 180$; $x + 4$

Using the remainder theorem, you can find the remainder. Suppose the value of the polynomial P when $x = a$ is zero. This means that 'the remainder when P is divided by $x - a$ is zero'. Therefore P is divisible by $x - a$. In other words $x - a$ is a factor of P .

This discussion leads us to the factor theorem, which is stated below.

Factor Theorem : If the value of a polynomial P in x when $x = a$ is zero, then $x - a$ is a factor of P .

Ex. 1: Find whether $x - 2$ is a factor of $2x^5 - 3x^3 + 4x^2 - 3x + 5$.

$$\text{Let } A = 2x^5 - 3x^3 + 4x^2 - 3x + 5$$

$$\therefore A(2) = 2(2)^5 - 3(2)^3 + 4(2)^2 - 3(2) + 5$$

$$= 64 - 24 + 16 - 6 + 5$$

$$= 85 - 30 = 55$$

$$\therefore A(2) \neq 0 \therefore x - 2 \text{ is not a factor of } A.$$

Instead of finding the value of a polynomial, it is more convenient to use synthetic division. By synthetic division, the calculations become simple and the remainder can be found easily. By this method, we cannot only decide whether $x - a$ is a factor or not, but also obtain the second factor. This will be clear from the following example.

Ex. 2: Find whether $x + 3$ is a factor of $4x^3 - x^2 - 29x + 30$.

$$\begin{array}{r|rrrr} -3 & 4 & -1 & -29 & 30 \\ & & -12 & 39 & -30 \\ \hline & 4 & -13 & 10 & \boxed{0} \end{array}$$

The remainder $= 0 \therefore x + 3$ is a factor. The other factor is $4x^2 - 13x + 10$. This is a quadratic polynomial and its factors are $(x - 2)$ and $(4x - 5)$.

$$\begin{aligned} \therefore 4x^3 - x^2 - 29x + 30 &= (x + 3)(4x^2 - 13x + 10) \\ &= (x + 3)(x - 2)(4x - 5). \end{aligned}$$

Ex. 3: If $x - 3$ is a factor of $x^4 - 13x^2 + px + 6$, find p .

$$\text{Let } P = x^4 - 13x^2 + px + 6$$

$$\therefore P(3) = (3)^4 - 13(3)^2 + p(3) + 6$$

$$= 81 - 117 + 3p + 6 = 3p - 30$$

$$\therefore x - 3 \text{ is a factor } \therefore P(3) = 0$$

$$\therefore 3p - 30 = 0 \therefore p = 10.$$

EXERCISE 7-13

In examples 1 to 15, find whether the second polynomial is a factor of the first. In case it is a factor, find the other factor.

1. $x^2 + 7x + 10$; $x + 2$
2. $7x^3 - 4x^2 - x - 2$; $x - 1$
3. $2x^4 + x^3 - x + 2$; $x - 4$
4. $3x^4 - 5x^3 + 4x^2 - 6x - 5$; $x + 1$
5. $2x^5 - 3x^3 + 8x^2 - 2x + 4$; $x + 2$
6. $5x^3 - 12x + 16$; $x + 2$
7. $3x^5 - 2x^3 + 8x + 9$; $x + 1$
8. $2x^5 - 3x^2 + x - 1$; $x - 2$
9. $3x^3 - 5x^2 + x - 7$; $x + 2$
10. $x^3 - 2x^2 + 5x - 10$; $x - 2$
11. $3x^3 + x^2 - 5x + 6$; $x + 2$
12. $x^3 - 2x^2 - x + 28$; $x - 4$
13. $x^3 - 3x^2 + 2x - 8$; $x + 4$
14. $3x^3 + 5x^2 + x - 1$; $x + 1$
15. $3x^4 + 3x^3 - 14x^2 - 8x + 16$; $x + 2$
16. If $x + 4$ is a factor of $x^2 + 9x + c$, find c .
17. $x + 6$ is a factor of $x^2 - 5x - p$, find p .
18. If $x + 2$ is a factor of $x^3 + 7x^2 + 7x - a$, find a .
19. Find p , if $x - 3$ is a factor of $x^2 + px + 6$.
20. If $x + 3$ is a factor of $3x^3 + ax - 12$, find a .
21. If $x + 1$ is a factor of $ax^2 - 2x - 7$, find a .
22. If $x - 1$ is a factor of $x^2 + (a + 2)x - 28$, find a .
23. If $x - 1$ is a factor of $x^2 + (p + 3)x - 20$, find p .
24. If $x + 2$ is a factor of $ax^3 + x^2 - 5x + a$, find a .
25. Find p and q if $x + 2$ and $x - 7$ are factors of the polynomial $3x^3 + px^2 - 47x + q$.

§ 7.12 Divisibility Tests :

By factor theorem, we can test whether $x - a$ is a factor of a given polynomial in x . But when the divisors are $x \pm 1$, the tests simplify further.

Test 1 : for $x - 1$.

Let us find the remainder when the polynomial $P = ax^4 + bx^3 + cx^2 + dx + e$ is divided by $x - 1$. When $x = 1$, $x^2 = 1$, $x^3 = 1$, $x^4 = 1 \dots \dots$ etc.

$$\therefore \text{remainder} = P(1) = a(1)^4 + b(1)^3 + c(1)^2 + d(1) + e \\ = a + b + c + d + e$$

If this remainder is zero, then $x - 1$ is a factor of P . We observe here that $a + b + c + d + e$ is the sum of the coefficients of the given polynomial. This gives us the following test for $x - 1$.

If the sum of the coefficients of a polynomial in x is 0, then $x - 1$ is a factor of that polynomial.

When it is ascertained that $x - 1$ is a factor, we can find the other factor by the procedure discussed below —

By synthetic division,

1	a	b	c	d	e
		a	$a + b$	$a + b + c$	$a + b + c + d$
	a	$a + b$	$a + b + c$	$a + b + c + d$	$a + b + c + d + e$

Here the remainder $= a + b + c + d + e = 0$ and the other factor is the quotient $ax^3 + (a + b)x^2 + (a + b + c)x + (a + b + c + d)$.

The degree of the given polynomial is 4. Hence if $x - 1$ is a factor of this polynomial, then the degree of the second factor will be 3 and it will contain 4 terms. How to write the coefficients of these four terms, will be clear to you, from the following table.

Term	coefficient	How to get this coefficient ?
First	a	Coefficient of x^4
Second	$a + b$	Sum of the coefficient of x^4 and x^3
Third	$a + b + c$	Sum of the coefficient of x^4 , x^3 and x^2
Fourth	$a + b + c + d$	Sum of the coefficient of x^4 , x^3 , x^2 and x .

Explanation : After writing the first term of the second factor add the first and second coefficients of the given polynomial. This sum is the coefficient of the second term.

To this second coefficient, add the third coefficient of the given polynomial. This sum is the coefficient of the third term.

To the third coefficient, add the fourth coefficient of the given polynomial. This sum is the coefficient of the fourth term.

Repeat the process till you get the constant term of the second factor.

In the above example, after having decided that $x - 1$ is a factor of—
 $4x^4 - 3x^3 + x^2 + x - 3$, the other factor can be written directly as follows.

$$4x^3 + (4 - 3)x^2 + (1 + 1)x + (2 + 1)$$

$$= 4x^3 + x^2 + 2x + 3$$

$$\therefore 4x^4 - 3x^3 + x^2 + x - 3 = (x - 1)(4x^3 + x^2 + 2x + 3).$$

EXERCISE 7.14

Find whether $x - 1$ is a factor of each of the following polynomials. If it is a factor, then write down the other factor directly.

1. $x^3 + 3x^2 - 5x + 1$

2. $x^3 + 2x^2 + 5x - 9$

3. $x^4 - 3x^3 + 4x^2 + 6x - 8$

4. $4x^4 - 7x^3 - 5x^2 + 11x - 3$

5. $7x^5 - 9x^4 + 8x^3 + 3x^2 - 11x + 2$

6. $5x^4 + 2x^3 - 9x + 2.$

Test 2 : for $x + 1$.

Let us find the remainder when the polynomial $P = ax^4 + bx^3 + cx^2 + dx + e$ is divided by $x + 1$.

If $x = -1$, $x^2 = 1$, $x^3 = -1$, $x^4 = 1$, $x^5 = -1$, etc.

$$\therefore \text{remainder} = P(-1) = a(-1)^4 + b(-1)^3 + c(-1)^2 + d(-1) + e$$

$$= a - b + c - d + e.$$

If this remainder is zero, then $x + 1$ is a factor of P .

$$\text{Now } a - b + c - d + e = 0 \therefore a + c + e = b + d.$$

Here a, c, e are the coefficients of even powers of x (i.e. of x^4, x^2, x^0) and b, d the coefficients of odd powers of x (i.e. of x^3, x). [e is a non-zero constant, therefore its degree is 0 which is an even integer]. This gives us the following test.

If for a given polynomial in x , the sum of the coefficients of even powers of x is equal to the sum of the coefficients of odd powers of x , then $x + 1$ is a factor of the given polynomial.

After having decided that $x + 1$ is a factor of a given polynomial, it is easy to obtain the second factor. The method is explained below.

By synthetic division

$$\begin{array}{r|rrrrrr}
 -1 & a & b & c & d & e & \\
 & & -a & -b+a & -c+b-a & -d+c-b+a & \\
 \hline
 & a & b-a & c-b+a & d-c+b-a & e-d+c-b+a &
 \end{array}$$

The remainder is zero $\therefore$ the second factor is the quotient
 $= ax^3 + (b-a)x^2 + (c-b+a)x + (d-c+b-a).$

This can be written as, $ax^3 + (b-a)x^2 + [c - (b-a)]x + [d - (c-b+a)]$.

After writing the first term ax^3 , subtract the coefficient a of this term from the second coefficient b of the given polynomial and write the second term as $(b-a)x^2$.

Subtract this new coefficient $(b-a)$ from the third coefficient c of the given polynomial and write the third term as $[c - (b-a)]x$.

Subtract this coefficient $(c-b+a)$ from the fourth coefficient d of the given polynomial and write the fourth term as $[d - (c-b+a)]$.

After a little practice, doing such subtractions, you can write the second factor directly.

Ex: Show that $x + 1$ is a factor of $7x^4 + 20x^3 + 3x^2 - 4x + 6$ and write down the second factor.

Sum of the coefficients of even powers of $x = 7 + 3 + 6 = 16$.

Sum of the coefficients of odd powers of $x = 20 - 4 = 16$.

$\therefore$ these sums are equal $\therefore x + 1$ is a factor.

The second factor is $7x^3 + (20-7)x^2 + (3-13)x + [-4 - (-10)]$
 $= 7x^3 + 13x^2 - 10x + 6.$

$\therefore 7x^4 + 20x^3 + 3x^2 - 4x + 6 = (x + 1)(7x^3 + 13x^2 - 10x + 6).$

EXERCISE 7-15 (Oral)

Find whether $x + 1$ is a factor of each of the following polynomials. If $x + 1$ is a factor, write the second factor.

- $x^3 + 3x^2 - 4x - 6$
- $x^3 + 5x^2 - 2x - 5$
- $x^4 - 3x^3 + 2x^2 + 7x + 1$
- $3x^4 - 2x^3 + 4x^2 + 12x + 3$
- $6x^5 + 9x^4 + 5x^3 + 7x^2 + 5x$
- $7x^4 - 9x^3 + 5x + 7$

Test 3 : for $x - p$.

Let us divide the polynomial $P = ax^2 + bx + c$ by $x - p$.

$$\begin{array}{r|rrrr}
 p & a & b & c & \\
 & & ap & ap^2 + bp & \\
 \hline
 & a & ap + b & ap^2 + bp + c &
 \end{array}$$

If the remainder $ap^2 + bp + c$ is 0 then $x - p$ is a factor. Here $ap^2 + bp + c$ is the value of the polynomial P when $x = p$. If this value is zero, then $x - p$ is a factor of the polynomial P . This discussion leads us to the following test.

If the value of a polynomial P , when x is replaced by p , is 0 then $x-p$ is a factor of P .

You might have noticed that, this in fact, is the factor theorem. Note that in using this test, sometimes the calculations become very lengthy and hence it is convenient to use synthetic division.

§ 7.13 Factorization :

The tests discussed above enable us to find factors of degree 1. Using them we wish to split a given polynomial as far as possible and express it as a product of polynomials which cannot be factored further. To express a polynomial in this form is called 'factorizing the given polynomial'.

As for the process of factorization we shall confine ourselves to polynomials whose coefficient of the highest degree term is 1. Such polynomials are called 'monic polynomials'. Polynomials which are not monic can also be factorized, but we are not going to consider such polynomials, in this course.

While factorizing a given polynomial, first test whether $x \pm 1$ are factors. After having exhausted the possible factors of the form $x \pm 1$, proceed as in the example discussed below.

Consider the polynomial $P = x^3 - 4x^2 - 11x + 30$. Applying the tests for $x \pm 1$, you can easily check that $x \pm 1$ are not the factors.

Now look at the constant term. Its divisors are 1, 2, 3, 5, 6, 10, 15, 30. It can be shown that the constant term of the polynomial is obtained as the product of the constant terms of all the factors of the polynomial.

$\therefore$ The first degree factors of P can be only from amongst $x \pm 1$; $x \pm 2$; $x \pm 3$; $x \pm 5$; $x \pm 6$; $x \pm 10$; $x \pm 15$; $x \pm 30$.

The following test will now enable us to eliminate those (amongst these) which are definitely not the factors of P .

If the sum of all the coefficients of a given polynomial is divisible by the sum of the coefficients of $x + a$ (i.e. $a + 1$), then $x + a$ may be a factor of the given polynomial i.e. if the sum of all the coefficients of a given polynomial is not divisible by $a + 1$, then $x + a$ cannot be a factor of that polynomial.

Now let us apply this test to the polynomial P in deciding which of the above 16 expressions are not the factors of P .

Sum of the coefficients of $P = 1 - 4 - 11 + 30 = 16$.

Sum of the coefficients of $(x + 2) = 1 + 2 = 3$.

Now 16 is not divisible by 3. $\therefore x + 2$ is not a factor of P .

Sum of the coefficients of $(x + 5) = 1 + 5 = 6$.

16 is not divisible by 6. $\therefore x + 5$ is not a factor of P .

Sum of the coefficients of $(x - 2) = 1 - 2 = -1$.

16 is divisible by -1 . $\therefore x - 2$ may be a factor.

In this way, by the above test, we see that $x + 2$; $x + 5$; $x \pm 6$; $x \pm 10$; $x - 15$; $x \pm 30$ are certainly not the factors. Having eliminated these, we can infer that $x - 2$; $x \pm 3$; $x - 5$; $x + 15$ may be factors.

Now use synthetic division to test $x - 2$; $x \pm 3$; $x - 5$; $x + 15$ for factors. Once we get the first factor, the second factor is also obtained. The second factor will be a quadratic polynomial. Its factors can be obtained by the method which you already know.

Now let us decide whether $x - 2$ is a factor or not.

$$\begin{array}{r|rrrr} 2 & 1 & -4 & -11 & 30 \\ & & 2 & -4 & -30 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

The remainder is zero. $\therefore x - 2$ is a factor and the other factor is $x^2 - 2x - 15$.

$$\begin{aligned} \therefore x^3 - 4x^2 - 11x + 30 \\ (x - 2)(x^2 - 2x - 15) \\ (x - 2)(x + 3)(x - 5). \end{aligned}$$

Ex. 1 : Factorize $x^3 - 14x^2 + 53x - 40$.

$$\begin{aligned} \text{Sum of the coefficients} &= 1 - 14 + 53 - 40 = 0 \\ \therefore x - 1 \text{ is a factor.} \end{aligned}$$

The other factor [as explained on page no. 128] is

$$\begin{aligned} x^2 + (1 - 14)x + (-13 + 53) \\ = x^2 - 13x + 40 \end{aligned}$$

$$\begin{aligned} \therefore x^3 - 14x^2 + 53x - 40 \\ = (x - 1)(x^2 - 13x + 40) = (x - 1)(x - 5)(x - 8). \end{aligned}$$

Ex. 2 : Factorize $x^4 + x^3 - 7x^2 - 13x - 6$.

You can verify that the sum of the coefficients $\neq 0$. $\therefore x - 1$ is not a factor. Sum of the coefficients of even powers of x

$$= 1 - 7 - 6 = -12.$$

Sum of the coefficients of odd powers of $x = 1 - 13 = -12$.

These sums are equal $\therefore x + 1$ is a factor. The other factor [as explained on page no. 129] is

$$\begin{aligned} x^3 + (1 - 1)x^2 + (-7 - 0)x + [-13 - (-7)] \\ = x^3 - 7x - 6 \text{ [or use synthetic division to get this factor].} \end{aligned}$$

For this factor

Sum of the coefficients of odd powers of $x = 1 - 7 = -6$.

Sum of the coefficients of even powers of $x = -6$. These sums are equal. $\therefore x + 1$ is a factor again.

The third factor [as explained on page no. 129] is

$$\begin{aligned} x^2 + (0 - 1)x + [-7 - (-1)] \\ = x^2 - x - 6 \quad \text{[or use synthetic division]} \end{aligned}$$

$$\begin{aligned} \therefore x^4 + x^3 - 7x^2 - 13x - 6 \\ = (x + 1)(x^3 - 7x - 6) \\ = (x + 1)(x + 1)(x^2 - x - 6) \\ = (x + 1)^2(x + 2)(x - 3). \end{aligned}$$

Ex. 3 : Factorize $x^4 + 6x^3 + 12x^2 + 11x + 6$.

All the coefficients of this polynomial are positive. Therefore their sum cannot be zero. $\therefore x - 1$ is not a factor.

Sum of the coefficients of even powers of $x = 1 + 12 + 6 = 19$.

Sum of the coefficients of odd powers of $x = 6 + 11 = 17$.

These sums are not equal. $\therefore x + 1$ is not a factor.

The constant term in the given polynomial is 6. Its divisors (except 1) are 2, 3, 6.

$x - 2, x - 3, x - 6$ cannot be the factors, for by replacing x by 2 or 3 or 6. The value of the polynomial will be a positive number and not zero.

Let us test whether $x + 2$ is a factor or not.

$$\begin{array}{r|rrrrr} -2 & 1 & 6 & 12 & 11 & 6 \\ & & -2 & -8 & -8 & -6 \\ \hline & 1 & 4 & 4 & 3 & 0 \end{array}$$

The remainder is 0. $\therefore x + 2$ is a factor, and the other factor is $x^3 + 4x^2 + 4x + 3$.

This cannot have $x \pm 1, x - 3$ as factors because these are not the factors of the given polynomial.

Let us therefore test whether $x + 3$ is a factor or not.

$$\begin{array}{r|rrrr} -3 & 1 & 4 & 4 & 3 \\ & & -3 & -3 & -3 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

The remainder is 0. $\therefore x + 3$ is a factor, and the third factor is $x^2 + x + 1$.

This cannot be factorized further.

$$\begin{aligned} \therefore x^4 + 6x^3 + 12x^2 + 11x + 6 \\ &= (x + 2)(x^3 + 4x^2 + 4x + 3) \\ &= (x + 2)(x + 3)(x^2 + x + 1). \end{aligned}$$

EXERCISE 7.16

Factorise the following polynomials.

1. $x^3 + 7x^2 + 7x - 15$
2. $x^3 + 6x^2 + 11x + 6$
3. $x^3 - 2x^2 - 5x + 6$
4. $x^3 - 3x + 2$
5. $x^3 - 7x + 6$
6. $x^3 - 6x^2 + 32$
7. $x^3 - 39x - 70$
8. $x^3 - 19x + 30$
9. $x^3 + 13x^2 + 10x - 24$
10. $x^3 + 4x^2 + 5x + 2$
11. $x^3 - 2x^2 + 3x - 2$
12. $x^3 - 3x^2 - 6x + 8$
13. $x^3 + 9x^2 + 23x + 15$
14. $x^3 + 6x^2 - 9x - 14$
15. $x^3 + 2x^2 - 11x - 12$
16. $x^3 + 6x^2 - x - 30$
17. $x^3 - 9x^2 + 26x - 24$
18. $x^3 - 14x^2 + 63x - 90$
19. $x^3 - 28x + 48$
20. $x^3 - 61x - 180$
21. $x^3 - 37x + 84$
22. $x^3 - 49x + 120$
23. $x^3 - 14x^2 + 61x - 84$
24. $x^3 - 10x^2 + 33x - 36$
25. $x^4 + x^3 + 2x^2 - x - 3$
26. $x^4 + 2x^3 + 4x^2 + 3x - 10$
27. $x^4 - 5x^3 + x^2 + 21x - 18$
28. $x^4 - 6x^3 + 9x^2 + 4x - 12$
29. $x^4 + 5x^3 + 5x^2 - 5x - 6$
30. $x^4 - x^3 - 3x^2 + x + 2$
31. $x^4 - 3x^3 + x^2 + 3x - 2$
32. $x^4 - 2x^3 + 2x^2 - 2x + 1$
33. $x^4 - 8x^3 + 17x^2 + 2x - 24$
34. $x^3 + x^2 - 36$
35. $x^3 + x^2 + 2x + 8$
36. $x^3 - 22x^2 + 129x - 108$
37. $x^3 + x^2 + 3x + 27$
38. $x^3 - 79x + 210$
39. $(x + 4)^3 - (x + 6)^2 + 8$
40. $x^4 + x^3 - 3x^2 - 5x - 2$
41. $x^4 - 6x^3 + 8x^2 + 6x - 9$
42. $x^4 + x^3 - x^2 - 7x - 6$
43. $x^4 - 2x^3 + x - 30$
44. $x^4 + 2x^3 - 2x^2 - 8x - 8$
45. $x^4 + 4x^3 + 6x^2 - 4x - 7$

§ 7.14 G. C. D. and L. C. M. :

(1) Greatest Common Divisor.

Consider the polynomials $(x - 1)^2 (x - 2)$ and $(x - 1)^2 (x - 2)^2$.

$A =$ Set of all divisors of $(x - 1)^2 (x - 2)$

$$= \{ 1, (x - 1), (x - 2), (x - 1)^2, (x - 1)(x - 2), (x - 1)^2 (x - 2) \}$$

$B =$ Set of all divisors of $(x - 1)^2 (x - 2)^2$.

$$= \{ 1, (x - 1), (x - 2), (x - 1)^2, (x - 2)^2, (x - 1)(x - 2), (x - 1)^2 (x - 2), (x - 1)(x - 2)^2, (x - 1)^2 (x - 2)^2 \}$$

$\therefore A \cap B =$ Set of all common divisors.

$$= \{ 1, (x - 1), (x - 2), (x - 1)^2, (x - 1)(x - 2), (x - 1)^2 (x - 2) \}$$

Of these common divisors, the divisor $(x - 1)^2 (x - 2)$ is a common divisor with the highest degree. This common divisor of highest degree is called the **greatest common divisor** (g. c. d.) of the given polynomials.

Recall that the greatest common divisor of two integers is the maximum (greatest) of the common divisors of the two integers.

Let us see how the greatest common divisor $(x - 1)^2 (x - 2)$ of the above polynomials is obtained. Consider the common divisors $(x - 1)$ and $(x - 2)$ of first degree separately. Decide which maximum power of $(x - 1)$ is contained in the common divisors. This is $(x - 1)^2$. Similarly decide which maximum power of $(x - 2)$ is contained in the common divisors. This is $(x - 2)^1$ i.e. $(x - 2)$. The product of these maximum powers viz. $(x - 1)^2 (x - 2)$ is the greatest common divisor of the given polynomials.

Ex. Find the greatest common divisor of the polynomials

$$(x - 3)^4 (x + 4)^2 (x + 5) \text{ and } (x - 3)^3 (x + 4)^4.$$

The common divisors (of first degree) of these polynomials are $(x - 3)$ and $(x + 4)$.

Let us decide which maximum power of $(x - 3)$ is contained in the common divisors of the given polynomials. The first polynomial contains $(x - 3)^4$. The second polynomial contains $(x - 3)^3$. $\therefore$ the common divisors will contain $(x - 3)$, $(x - 3)^2$, $(x - 3)^3$ as powers of $(x - 3)$. The maximum power of $(x - 3)$ contained in common divisors is $(x - 3)^3$.

Similarly the common divisors will contain $(x + 4)$ and $(x + 4)^2$ as powers of $(x + 4)$.

The maximum power of $(x + 4)$ contained in common divisors is $(x + 4)^2$.

$$\therefore \text{g. c. d.} = (x - 3)^3 (x + 4)^2.$$

Note.—The first polynomial contains $(x - 3)^4$ and the second polynomial contains $(x - 3)^3$. $(x - 3)^3$ divides both the polynomials but $(x - 3)^4$ does not divide the second i.e. the maximum power of $(x - 3)$ which divides both the polynomials is $(x - 3)^3$. Therefore $(x - 3)^3$ is taken as a divisor in the greatest common divisor. Similarly $(x + 4)^2$ divides both the polynomials but $(x + 4)^3$ does not divide the first i.e. the maximum power of $(x + 4)$ which divides both the polynomials is $(x + 4)^2$. Therefore $(x + 4)^2$ is taken as the other divisor in the greatest common divisor.

This discussion leads us to the procedure of finding the greatest common divisor of two or more polynomials. Factorise all the given polynomials. Consider every

common divisor (in first degree) separately. Take the maximum power of each of these, which divides all the polynomials. The product of all such maximum powers of common divisors is the required greatest common divisor of the given polynomials.

EXERCISE 7-17

Write down the greatest common divisors of the following polynomials.

1. $(x-1)^3(x-2)^2$; $(x-1)^3(x-2)^3(x+3)$
2. $(x+2)^4(x-2)^2(x+5)^3$; $(x+2)^5(x-2)(x+5)^2$
3. $(x+3)(x-4)^3(x-2)^2$; $(x+3)^2(x-4)^2(x+4)$
4. $(x+1)(x-2)^5(x+3)^4$; $(x+2)(x-2)^4(x+5)$
5. $x^2(x-1)^4(x+2)(x-3)^3$; $x^3(x+2)^3(x-3)^2$

(2) Least Common Multiple.

$(x+1)$, $(x+1)^2$, $(x+1)^3$, ... $(x+1)^n$, ... are all divisible by $(x+1)$. They are called **multiples** of $(x+1)$.

$(x+1)^2(x-2)$, $(x+1)^3(x-2)$, $(x+1)^2(x-2)^2$, $(x+1)^3(x-2)^2$, $(x+1)^3(x-2)^3$, $(x+1)^4(x-2)^2$, etc. are all multiples of $(x+1)^2(x-2)$.

Now consider the polynomials $(x+1)^3(x-2)^2$ and $(x+1)^2(x-2)^4$.

A = Set of all multiples of $(x+1)^3(x-2)^2$

$$= \{ (x+1)^3(x-2)^2, (x+1)^4(x-2)^2, (x+1)^5(x-2)^2, \dots \\ (x+1)^3(x-2)^3, (x+1)^4(x-2)^3, (x+1)^5(x-2)^3, \dots \\ (x+1)^3(x-2)^4, (x+1)^4(x-2)^4, (x+1)^5(x-2)^4, \dots \}$$

B = Set of all multiples of $(x+1)^2(x-2)^4$

$$= \{ (x+1)^2(x-2)^4, (x+1)^3(x-2)^4, (x+1)^4(x-2)^4, \dots \\ (x+1)^2(x-2)^5, (x+1)^3(x-2)^5, (x+1)^4(x-2)^5, \dots \\ (x+1)^2(x-2)^6, (x+1)^3(x-2)^6, (x+1)^4(x-2)^6, \dots \}$$

A $\cap$ B = Set of all common multiples

$$= \{ (x+1)^3(x-2)^4, (x+1)^4(x-2)^4, (x+1)^5(x-2)^4, \dots \\ (x+1)^3(x-2)^5, (x+1)^4(x-2)^5, (x+1)^5(x-2)^5, \dots \\ (x+1)^3(x-2)^6, (x+1)^4(x-2)^6, (x+1)^5(x-2)^6, \dots \}$$

Of these common multiples, the multiple $(x+1)^3(x-2)^4$ is a common multiple of lowest degree. This common multiple of lowest degree is called the **least common multiple (l.c.m.)** of the given polynomials.

Recall that the least common multiple of two integers is the (minimum) least of the common multiples of the two integers.

Let us now see how the least common multiple $(x+1)^3(x-2)^4$ is obtained. The given polynomials contain $(x+1)^2$ and $(x+1)^3$ as powers of $(x+1)$ and $(x-2)^4$ and $(x-2)^3$ as powers of $(x-2)$. Consider all positive integral powers of $(x+1)$. Decide the least power amongst these, which is a multiple of the powers of $(x+1)$ contained in the polynomials, as factors. This least power is $(x+1)^3$.

Similarly consider all positive integral powers of $(x-2)$. Decide the least power amongst these, which is a multiple of the powers of $(x-2)$ contained in the polynomials as factors. This least power is $(x-2)^4$.

The product of these least powers is the desired least common multiple of the given polynomials.

Ex. : Find the least common multiple of $(x - 3)^4 (x + 4)^2 (x + 5)$ and $(x - 3)^3 (x + 4)^3$.

The given polynomials contain powers of $(x - 3)$, $(x + 4)$ and $(x + 5)$.

The least power of $(x - 3)$ which is a multiple of the powers of $(x - 3)$ contained in the polynomials as factors is $(x - 3)^4$.

The least power of $(x + 4)$ which is a multiple of the powers of $(x + 4)$ contained in the polynomials as factors is $(x + 4)^3$.

The least power of $(x + 5)$ which is a multiple of the powers of $(x + 5)$ contained in the polynomials as factors is $(x + 5)$.

The product of these least powers is $(x - 3)^4 (x + 4)^3 (x + 5)$.

$\therefore$ L. C. M. = $(x - 3)^4 (x + 4)^3 (x + 5)$.

EXERCISE 7.18

Write down the least common multiple of the polynomials in each of the following examples.

1. $(x - 1)^2 (x - 2)^2$; $(x - 1)^2 (x - 2)^3 (x + 3)$.
2. $(x + 2)^4 (x - 2)^2 (x + 5)^3$; $(x + 2)^5 (x - 2) (x + 5)^2$.
3. $(x + 3) (x - 4)^3 (x - 2)^2$; $(x + 3)^2 (x - 4)^2 (x + 4)$.
4. $(x + 1) (x - 2)^5 (x + 3)^4$; $(x + 2) (x - 2)^4 (x + 5)$.
5. $x^2 (x - 1)^4 (x + 2) (x - 3)$; $x^3 (x + 2)^3 (x - 3)^2$.

Find the G. C. D. and L. C. M. of the polynomials in each of the following examples.

1. $x^2 - 1$; $x^2 + 3x - 4$; $x^3 - 1$.
2. $x^3 + 5x + 6$; $x^2 - 9$; $x^4 + 27x$.
3. $x^3 + 6x^2 + 11x + 6$; $x^3 - 7x - 6$; $x^3 + x^2 - 9x - 9$.

Solution :

$$1. \quad x^2 - 1 = (x - 1) (x + 1)$$

$$x^2 + 3x - 4 = (x - 1) (x + 4)$$

$$x^3 - 1 = (x - 1) (x^2 + x + 1)$$

$$\therefore \text{G. C. D.} = x - 1; \text{L. C. M.} = (x - 1) (x + 1) (x + 4) (x^2 + x + 1).$$

$$2. \quad x^2 + 5x + 6 = (x + 3) (x + 2)$$

$$x^2 - 9 = (x + 3) (x - 3)$$

$$x^4 + 27x = x (x + 3) (x^2 - 3x + 9)$$

$$\therefore \text{G. C. D.} = x + 3; \text{L. C. M.} = x (x + 3) (x + 2) (x - 3) (x^2 - 3x + 9).$$

$$3. \quad x^3 + 6x^2 + 11x + 6 = x^3 + x^2 + 5x^2 + 5x + 6x + 6$$

$$= x^2 (x + 1) + 5x (x + 1) + 6 (x + 1)$$

$$= (x + 1) (x^2 + 5x + 6)$$

$$= (x + 1) (x + 2) (x + 3)$$

$$x^3 - 7x - 6 = x^3 + 1 - 7x - 7$$

$$= (x + 1) (x^2 - x + 1) - 7 (x + 1)$$

$$= (x + 1) (x^2 - x + 1 - 7)$$

$$= (x + 1) (x^2 - x - 6)$$

$$= (x + 1) (x + 2) (x - 3)$$

$$x^3 + x^2 - 9x - 9 = x^2 (x + 1) - 9 (x + 1)$$

$$= (x + 1) (x^2 - 9)$$

$$= (x + 1) (x + 3) (x - 3)$$

$$\therefore \text{G. C. D.} = x + 1; \text{L. C. M.} = (x + 1) (x + 2) (x + 3) (x - 3).$$

EXERCISE 7-19

Find the G. C. D. and L. C. M. of the following polynomials.

1. $x^2 + 7x + 12$; $x^3 + 4x^2 + 4x + 3$.
2. $x^3 - 6x^2 + 11x - 6$; $x^3 - 1$.
3. $x^3 + x - 2$; $x^4 + 3x^2 + 4$.
4. $4x^2 - 25$; $6x^2 - 23x + 20$.
5. $x^3 + 2x^2 - x - 2$; $x^3 + x^2 - 4x - 4$.
6. $4x^4 - 5x^2 + 1$; $4x^4 + 4x^3 + x^2 - 1$.
7. $x^2 + 5x + 6$; $x^2 + 7x + 12$; $x^2 + 9x + 20$.
8. $x^2 + 9x + 20$; $x^2 + 7x + 12$; $x^3 + 64$.
9. $x^3 + 1$; $x^4 + x^2 + 1$; $x^4 - 2x^3 + x^2 - 1$.
10. $2x^2 - 5x + 2$; $3x^2 - 8x + 4$; $x^4 - 8x$.
11. $4x^4 + 16x^3 - 20x^2$; $3x^3 + 14x^2 - 5x$; $x^4 + 125x$.
12. $x^4 - 3x^3 - 28x^2$; $3x^2 - 147$; $x^3 - 9x^2 + 14x$.
13. $4x^4 - 28x^3 + 48x^2$; $18x^3 - 162x$; $6x^4 - 162x$.
14. $x^3 + 4x^2 - 6x - 24$; $x^3 - 6x^2 - 6x + 36$; $x^2 - 2x - 24$.
15. $3x^5 - 81x^2$; $5x^3 - 45x$; $7x^3 - 42x^2 + 63x$.
16. $2x^2 - 5x + 2$; $x^3 - 4x$; $3ax - 2x + 4 - 6a$.
17. $72x^4 + 243x$; $24x^3 - 54x$; $18x^4 + 15x^3 - 18x^2$.
18. $3x^2(x^2 - 7x + 12)$; $18(x^3 - 8x^2 + 15x)$; $9x^3(x^2 - 4x + 3)$.
19. $2x^3 + 250$; $3x^2 - 75$; $4x^2 - 20x - 200$.
20. $6x^2 - 54$; $3x^3 + 6x^2 - 45x$; $9x^2 - 99x + 216$.
21. $12x^3 - 27x$; $12x^2 - 6x - 18$; $4x^3 - 4x^2 - 3x$.
22. $x^2 + 11x + 28$; $x^2 - x - 56$; $x^3 + 343$.
23. $2x^4 - 6x^3 - 56x^2$; $3x^2 - 147$; $x^3 - 9x^2 + 14x$.
24. $3x^2 + 16x - 35$; $x^3 + 343$; $2x^2 - 98$.
25. $x^3 - 4x$; $3x(x^2 - x - 6)$; $6x(x^3 + 8)$.
26. $40x^4 + 135x$; $12x^3 - 27x$; $2x^4 - x^3 - 6x^2$.
27. $2ax^3 - 8ax$; $6ax^3 - 48a$; $3ax^2 - 15ax + 18a$.
28. $6x^3 + 18x^2 - 60x$; $4x^2 - 28x + 40$; $8x^2 - 40x + 48$.
29. $(x + 3)^2 - 9x - 27$; $x^3 + x^2 - 6x$; $x^5 - 13x^3 + 36x$.
30. $x^3 - 7x + 6$; $x^3 + 2x^2 - 5x - 6$; $x^4 + 3x^3 - 8x - 24$.
31. $x^3 - 49x - 120$; $x^3 + 7x^2 + 7x - 15$; $x^3 + 9x^2 + 23x + 15$.
32. $x^2 + 3x + 2$; $x^4 - 5x^2 + 4$; $(x^2 - 2x)^2 - 11(x^2 - 2x) + 24$.
33. $2x^6 - 128$; $x^4 + 4x^2 + 16$; $2x^2 + 4x + 8$.
34. $2x^4 + 8$; $x^3 - 2x + 4$; $x^4 - 4x^3 + 4x^2 - 4$.
35. $x^4 - 13x^2 + 36$; $5x^3 - 40$; $5x^2 + 5x - 30$.
36. $x^4 - 29x^2 + 100$; $x^4 - 5x^2 + 4$; $3x^4 - 48$.
37. $27x^3 + 1$; $87x^2 + 8x - 7$; $27x^3 + 27x^2 + 9x + 1$.
38. $x^2 - 4$; $x^3 + 8$; $x^3 + 4x^2 + 8x + 8$.
39. $x^4 + x^2 + 1$; $x^4 - x$; $(x - 1)^2 + 3x$.
40. $x^4 - 1$; $x^6 - 1$; $x^8 - 1$.

Chapter 8 : RATIONAL ALGEBRAIC EXPRESSIONS

§ 8.1 Revision :

You have already learnt that expressions of the type $\frac{x^2+5}{x-2}$; $\frac{1}{x^2-2x-3}$; $\frac{x-4}{x^3-64}$; they are called rational algebraic expressions. If A and B are polynomials and B is a non-zero polynomial then the ratio $\frac{A}{B}$ is called a rational algebraic expression. In order that a rational algebraic expression should be meaningful, the polynomial in the denominator should not be zero.

We have used some properties of rational algebraic expressions to simplify them. They are stated below.

Cancellation Property :

(1) If A, B, K are polynomials and $B \neq 0$, $K \neq 0$, then

$$\frac{A \times K}{B \times K} = \frac{A}{B}.$$

(2) If A, B are polynomials and $B \neq 0$, then

$$(i) -\left(\frac{A}{B}\right) = -\frac{A}{B} = \frac{-A}{B} = \frac{A}{-B}$$

$$\text{and } (ii) \frac{A}{B} = -\frac{-A}{B} = -\frac{A}{-B}$$

(3) If A, B, C, D are polynomials and $B \neq 0$, $D \neq 0$ then

$$(i) \frac{A}{B} \times \frac{C}{D} = \frac{A \times C}{B \times D} \quad (ii) \frac{A}{B} \div \frac{C}{D} = \frac{A \times D}{B \times C} \quad (C \neq 0)$$

(4) If A, B, C, D are polynomials and $B \neq 0$, $D \neq 0$, then

$$(i) \frac{A}{B} + \frac{C}{B} = \frac{A+C}{B} \quad (ii) \frac{A}{B} - \frac{C}{B} = \frac{A-C}{B}$$

$$(iii) \frac{A}{B} + \frac{C}{D} = \frac{A \times D + B \times C}{B \times D} \quad (iv) \frac{A}{B} - \frac{C}{D} = \frac{A \times D - B \times C}{B \times D}$$

To revise these properties, solve the following exercise (8.1).

EXERCISE 8.1

Simplify.

$$1. \frac{12x^3y^4z^2}{18x^4y^3}$$

$$2. \frac{74x^3a^4b^3}{111x^2yz}$$

$$3. \frac{3x^2-6xy}{x^2-4y^2}$$

$$4. \frac{a^3-ab^2}{a^3+2a^2b+ab^2}$$

$$5. 1-a+\frac{a^2}{1+a}$$

$$6. \frac{2x^2+5xy-3y^2}{2x^2-7xy+3y^2}$$

- $$\begin{array}{lll}
7. \frac{3x^2 - 19x - 14}{6x^2 - 11x - 10} & 8. \frac{5x^2 - 4xy - 12y^2}{x^4 - 8xy^3} & 9. \frac{a^2 + a - 12}{a^3 + 5a - 42} \\
10. \frac{x + y}{x^2 - y^2} \times \frac{x^2 - 2xy + y^2}{x^2 - xy} & 11. \frac{x^2 - 3x + 2}{x^2 + x - 12} \times \frac{x^2 + 6x + 8}{x^2 - 4} & \\
12. \frac{a^2 + 3a}{a^2 + 4a + 3} \times \frac{a^2 - 2a - 3}{a^2 - 9} & 13. \frac{3x^2 - 2x - 8}{4x^2 + 5x - 6} \times \frac{8x^2 + 6x - 9}{6x^2 + 17x + 12} & \\
14. \frac{4x^2 - 15x - 4}{3x^2 - x - 4} \div \frac{4x^2 - 3x - 1}{3x^2 - 7x + 4} & 15. \frac{3a^2 + a - 14}{2a^2 - a - 6} \div \frac{6a^2 + 5a - 21}{4a^2 - 9} & \\
16. \frac{3x^2 + x - 10}{2x^2 - 3x - 9} \div \frac{12x^2 - 29x + 15}{8x^2 + 6x - 9} & 17. \frac{x^2 - 3xy + 2y^2}{x^2 + xy - 2y^2} \div \frac{x^2 - xy - 2y^2}{x^2 + 5xy + 6y^2} & \\
18. \frac{2x^2 + 5x - 3}{3x^2 - x - 2} \times \frac{x^2 - 1}{x^2 + x - 6} \times \frac{3x^2 - 4x - 4}{2x^2 + x - 1} & & \\
19. \frac{3x + 2}{5x^2 - x} \times \frac{2x^2 - x}{2x^2 - x - 1} \times \frac{10x^2 + 3x - 1}{6x^2 + x - 2} & & \\
20. \frac{x^2 + x - 12}{x^2 - 5x + 6} \times \frac{x^2 - 25}{x^2 - x - 20} \div \frac{x^2 + 2x - 15}{x - 2} & & \\
21. \frac{x^2 - 64}{x^2 + 24x + 128} \div \frac{x^2 - 16}{x^2 + 12x - 64} \times \frac{x + 4}{x^2 - 16x + 64} & & \\
22. \frac{x^2 - xy}{xy + y^2} \div \left(\frac{x^2 - y^2}{x^2 + 2xy + y^2} \div \frac{x^2 - 2xy + y^2}{x^2y + xy^2} \right) & & \\
23. \left(\frac{x^2 - xy}{xy + y^2} \div \frac{x^2 - y^2}{x^2 + 2xy + y^2} \right) \div \frac{x^2 - 2xy + y^2}{x^2y + xy^2} & & \\
24. \frac{1}{x + 3} + \frac{1}{x + 2} & 25. \frac{2x - 13}{x - 4} - \frac{2x + 5}{x - 3} & 26. \frac{1}{x + 2} + \frac{2}{x^2 - 4} \\
27. \frac{x}{x - 3} - \frac{x^2}{x^2 - 9} & 28. \frac{a^2 - 9}{a^2 + a - 12} - \frac{a^2 + 6a + 5}{a^2 + 7a + 10} & 29. 1 - x + x^2 - \frac{x^3}{1 + x} \\
30. \frac{x - 3y}{x^2 - xy - 6y^2} - \frac{3x + 2y}{2x^2 - xy - 15y^2} + \frac{x + 2y}{2x^2 + 9xy + 10y^2} & &
\end{array}$$

§ 8.2 Compound Algebraic Expressions :

We have seen that a rational algebraic expression can be written in the form of the ratio of two polynomials. One polynomial is in the numerator and the other is in the denominator. This is like rational numbers. A rational number can be written in the form of the ratio of two integers.

Now consider the following numbers.

$$\frac{\frac{2}{5}}{3} \quad \frac{2}{\frac{5}{3}} ;$$

$$\frac{\frac{2}{5}}{3} = \frac{2}{5} \div \frac{3}{1} = \frac{2}{5} \times \frac{1}{3} = \frac{2}{15}$$

$$\text{and } \frac{2}{\frac{5}{3}} = \frac{2}{1} \div \frac{5}{3} = \frac{2}{1} \times \frac{3}{5} = \frac{6}{5}$$

$\therefore$ These are different rational numbers. They are written in the form of the ratio of two rational numbers. The numerator of the first number is $\frac{2}{5}$ and its denominator is 3, whereas the numerator of the second number is 2 and its denominator is $\frac{5}{3}$. In order to avoid confusion and ambiguity in such cases, we use a heavy or thick or a slightly longer line to show the numerator and denominator separately.

There is a close similarity of form between the rational algebraic expressions and the rational numbers. A rational algebraic expression has a numerator and a (non-zero) denominator. As in the case of the rational numbers considered above, the numerator and the denominator of a rational algebraic expression can be themselves rational algebraic expressions. e.g.

$$\frac{x+4}{\frac{3x-2}{5x+3}} ;$$

$$\frac{\frac{x+4}{3x-2}}{5x+3} ;$$

$$\frac{\frac{3x+4}{2x-1}}{\frac{4x+7}{x-2}} .$$

These expressions are similar to the rational numbers, described above. We call such expressions 'compound algebraic expressions'. Now let us see some methods of simplifying such compound algebraic expressions.

$$\frac{x+4}{\frac{3x-2}{5x+3}} = \frac{x+4}{3x-2} \div \frac{5x+3}{1} = \frac{x+4}{3x-2} \times \frac{1}{5x+3} = \frac{x+4}{(3x-2)(5x+3)}$$

$$\frac{\frac{x+4}{3x-2}}{5x+3} = \frac{x+4}{1} \div \frac{3x-2}{5x+3} = \frac{x+4}{1} \times \frac{5x+3}{3x-2} = \frac{(x+4)(5x+3)}{3x-2}$$

$$\frac{\frac{3x+4}{2x-1}}{\frac{4x+7}{x-2}} = \frac{3x+4}{2x-1} \div \frac{4x+7}{x-2} = \frac{3x+4}{2x-1} \times \frac{x-2}{4x+7} = \frac{(3x+4)(x-2)}{(2x-1)(4x+7)}$$

From this we can state the following rule.

If A, B, C, D are rational algebraic expressions and $B \neq 0$, $C \neq 0$, $D \neq 0$, then

$$\frac{\frac{A}{B}}{\frac{C}{D}} = \frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C} = \frac{A \times D}{B \times C} .$$

Ex. 1: Simplify.

$$\frac{\frac{3x+4}{x-3} + 4}{\frac{4x+5}{x-3} - 3}$$

$$\begin{aligned}\text{Numerator} &= \frac{3x+4}{x-3} + 4 = \frac{(3x+4) + 4(x-3)}{x-3} \\ &= \frac{3x+4+4x-12}{x-3} = \frac{7x-8}{x-3}\end{aligned}$$

$$\begin{aligned}\text{Denominator} &= \frac{4x+5}{x-3} - 3 = \frac{(4x+5) - 3(x-3)}{x-3} \\ &= \frac{4x+5-3x+9}{x-3} = \frac{x+14}{x-3}\end{aligned}$$

$\therefore$ Expression = Numerator $\div$ Denominator

$$\begin{aligned}&= \frac{7x-8}{x-3} \div \frac{x+14}{x-3} = \frac{7x-8}{x-3} \times \frac{x-3}{x+14} \\ &= \frac{7x-8}{x+14} \quad (x \neq 3; x \neq -14).\end{aligned}$$

Ex. 2: Simplify.

$$\frac{\frac{4x-1}{2x+3} - 3}{4 - \frac{3x+5}{x+2}}$$

$$\begin{aligned}\text{Numerator} &= \frac{4x-1}{2x+3} - 3 = \frac{(4x-1) - 3(2x+3)}{2x+3} \\ &= \frac{4x-1-6x-9}{2x+3} = \frac{-2x-10}{2x+3} = \frac{-2(x+5)}{2x+3}\end{aligned}$$

$$\begin{aligned}\text{Denominator} &= 4 - \frac{3x+5}{x+2} = \frac{4(x+2) - (3x+5)}{x+2} \\ &= \frac{4x+8-3x-5}{x+2} = \frac{x+3}{x+2}\end{aligned}$$

$\therefore$ Expression = Numerator $\div$ Denominator

$$\begin{aligned}&= \frac{-2(x+5)}{2x+3} \div \frac{x+3}{x+2} = \frac{-2(x+5)}{2x+3} \times \frac{x+2}{x+3} \\ &= \frac{-2(x+5)(x+2)}{(2x+3)(x+3)}\end{aligned}$$

Ex. 3 : If $y = \frac{2x+5}{3x-7}$ express the expression $\frac{2y+3}{3y-2}$ in x .

Writing $\frac{2x+5}{3x-7}$ for y , $\frac{2y+3}{3y-2}$ becomes $\frac{2\left(\frac{2x+5}{3x-7}\right)+3}{3\left(\frac{2x+5}{3x-7}\right)-2}$.

$$\begin{aligned}\text{Numerator} &= 2\left(\frac{2x+5}{3x-7}\right)+3 = \frac{2(2x+5)+3(3x-7)}{3x-7} \\ &= \frac{4x+10+9x-21}{3x-7} = \frac{13x-11}{3x-7}\end{aligned}$$

$$\begin{aligned}\text{Denominator} &= 3\left(\frac{2x+5}{3x-7}\right)-2 = \frac{3(2x+5)-2(3x-7)}{3x-7} \\ &= \frac{6x+15-6x+14}{3x-7} = \frac{29}{3x-7}\end{aligned}$$

$\therefore$ The given expression = Numerator $\div$ Denominator

$$\begin{aligned}&= \frac{13x-11}{3x-7} \div \frac{29}{3x-7} = \frac{13x-11}{3x-7} \times \frac{3x-7}{29} \\ &= \frac{13x-11}{29}\end{aligned}$$

EXERCISE 8.2

Simplify.

1. $\frac{\frac{7}{2}+4}{\frac{5}{2}}$

2. $\frac{\frac{5}{9}}{\frac{3}{7}+1}$

3. $\frac{\frac{3x+2}{5x}}{2}$

4. $\frac{\frac{7}{x+4}}{3}$

5. $\frac{\frac{7}{x+4}}{3}$

6. $\frac{\frac{3x+2}{5x}}{2}$

7. $\frac{\frac{x+4}{x-2}}{2x-1}$

8. $\frac{\frac{x+4}{x-2}}{2x-1}$

9. $\frac{\frac{7x+5}{2x-3}}{5x-4}$

10. $\frac{x+\frac{1}{x}}{x-\frac{1}{x}}$

11. $\frac{x+\frac{1}{x}+2}{x+\frac{1}{x}-2}$

12. $\frac{\frac{3x+x^3}{1+3x^2}+1}{\frac{3x+x^3}{1+3x^2}-1}$

13. $\frac{\frac{x-4}{4x-1}-4}{4\left(\frac{x-4}{4x-1}\right)-1}$

14. $\frac{\frac{x+2}{2x-1}+2}{2\left(\frac{x+2}{2x-1}\right)-1}$

15. $\frac{\frac{3+5x}{4x-1}+3}{4\left(\frac{3+5x}{4x-1}\right)-5}$

16. $\frac{2\left(\frac{2x+1}{3x-2}\right)+1}{3\left(\frac{2x+1}{3x-2}\right)-2}$

17. $\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$

18. $\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

19. $\frac{x+3-\frac{4}{x}}{x+9+\frac{20}{x}}$ 20. $\frac{a-\frac{a-b}{1+ab}}{1+\frac{a(a-b)}{1+ab}}$ 21. $\frac{\frac{1}{a-b}-\frac{1}{a+b}}{\frac{a+b}{a-b}+\frac{b-a}{a+b}}$
22. $\frac{1}{1+\frac{c}{a+b}} + \frac{1}{1+\frac{a}{b+c}} + \frac{1}{1+\frac{b}{c+a}}$
23. If $y = \frac{x-1}{x+1}$, express $\frac{y-1}{y+1}$ in x .
24. If $y = \frac{x-4}{4x-1}$, express $\frac{y-4}{4y-1}$ in x .
25. If $y = \frac{3+5x}{4x-1}$, express $\frac{y+3}{4y-5}$ in x .
26. If $y = \frac{2x+1}{3x-2}$, express $\frac{2y+1}{3y-2}$ in x .
27. If $x = 99$, find the value of $\left(x^3 + \frac{1}{x}\right) \left(\frac{1}{1-x^2}\right) - \frac{1}{x^2+x} + \frac{x}{x-1}$.
28. If $x = 101$, find the value of $\frac{3x+1}{x^4-1} - \frac{1}{2(x+1)} + \frac{3x+1}{2(x^2+1)}$.
29. If $x = 99, y = 100$, find the value of $\left(x-1+\frac{1}{x}\right) \left(y^3-1\right) \div \left(y+1+\frac{1}{y}\right) \left(x^3+1\right)$.
30. If $x = 100, y = 98$, find the value of $\left(\frac{x^3}{1-x} + 1 + x + x^2\right) \div \left(1-y+y^3-\frac{y^3}{1+y}\right)$.

§ 8.3 Equations :

You have already studied how to solve equations in one rational algebraic expressions. Now we shall solve some more equations of this type.

First Type:

Ex. 1: Solve. $\frac{3x+2}{3} - \frac{x-1}{x+4} = \frac{2x+1}{2}$

Part 1: $\frac{3x+2}{3} - \frac{x-1}{x+4} = \frac{2x+1}{2} \therefore \frac{3x+2}{3} - \frac{2x+1}{2} = \frac{x-1}{x+4}$

$\therefore \frac{2(3x+2)-3(2x+1)}{6} = \frac{x-1}{x+4} \therefore \frac{6x+4-6x-3}{6} = \frac{x-1}{x+4}$

$\therefore \frac{1}{6} = \frac{x-1}{x+4} \therefore x+4 = 6(x-1) \therefore x+4 = 6x-6$

$\therefore 10 = 5x \therefore x = 2$

Part 2: Putting $x = 2$ in the given equation

$$\text{L. H. S.} = \frac{3(2) + 2}{3} - \frac{(2) - 1}{(2) + 4} = \frac{8}{3} - \frac{1}{6} = \frac{16 - 1}{6} = \frac{15}{6} = \frac{5}{2}$$

$$\text{R. H. S.} = \frac{2(2) + 1}{2} = \frac{5}{2}$$

$$\therefore \text{Solution set} = \{2\}$$

or Putting $x = 2$ in the given equation

$$\frac{3(2) + 2}{3} - \frac{(2) - 1}{(2) + 4} \stackrel{?}{=} \frac{2(2) + 1}{2} \therefore \frac{8}{3} - \frac{1}{6} \stackrel{?}{=} \frac{5}{2}$$

$$\therefore \frac{5}{2} \stackrel{\checkmark}{=} \frac{5}{2}$$

Ex. 2: Solve. $\frac{2x + 8\frac{1}{2}}{9} - \frac{13x - 2}{17x - 32} = \frac{x}{4} - \frac{x + 16}{36}$

Part 1: $\frac{2x + 8\frac{1}{2}}{9} - \frac{13x - 2}{17x - 32} = \frac{x}{4} - \frac{x + 16}{36}$

$$\therefore \frac{4x + 17}{18} - \frac{x}{4} + \frac{x + 16}{36} = \frac{13x - 2}{17x - 32}$$

$$\therefore \frac{2(4x + 17) - 9x + (x + 16)}{36} = \frac{13x - 2}{17x - 32}$$

$$\therefore \frac{8x + 34 - 9x + x + 16}{36} = \frac{13x - 2}{17x - 32}$$

$$\therefore \frac{50}{36} = \frac{13x - 2}{17x - 32} \therefore \frac{25}{18} = \frac{13x - 2}{17x - 32}$$

$$\therefore 25(17x - 32) = 18(13x - 2) \therefore 425x - 800 = 234x - 36$$

$$\therefore 191x = 764 \therefore x = 4.$$

Part 2: Putting $x = 4$ in the given equation

$$\frac{2(4) + 8\frac{1}{2}}{9} - \frac{13(4) - 2}{17(4) - 32} \stackrel{?}{=} \frac{(4)}{4} - \frac{(4) + 16}{36}$$

$$\frac{8 + 17}{9} - \frac{52 - 2}{68 - 32} \stackrel{?}{=} 1 - \frac{20}{36} \therefore \frac{16 + 17}{18} - \frac{50}{36} \stackrel{?}{=} \frac{36 - 20}{36}$$

$$\therefore \frac{2(33) - 50}{36} \stackrel{?}{=} \frac{16}{36} \therefore \frac{66 - 50}{36} \stackrel{?}{=} \frac{16}{36} \therefore \frac{16}{36} \stackrel{\checkmark}{=} \frac{16}{36}$$

$$\therefore \text{Solution set} = \{4\}.$$

Remarks: Putting $x = 4$ in the L. H. S. and R. H. S. of the given equation separately, we can show that they are equal and in this way also we can verify that $x = 4$ is a solution of the given equation.

EXERCISE 8-3

Solve.

$$1. \frac{7x}{3} - [3x - \frac{1}{2}(2x - 3)] = \frac{1}{3} - 2x \quad 2. \frac{4x + 3}{3} + \frac{3(29 - 7x)}{12 - 5x} = \frac{8x + 19}{6}$$

$$\begin{aligned}
3. \quad \frac{10x+41}{18} - \frac{12x+38}{13x+23} &= \frac{5x+8}{9} & 4. \quad \frac{14x+5}{18} - \frac{5x-4}{3x-2} &= \frac{7x-11}{9} \\
5. \quad \frac{2x+4\frac{1}{2}}{5} - \frac{3x+5}{5x-25} &= \frac{2x}{5} & 6. \quad \frac{7x-8}{10} + \frac{5x-2}{2x+7} &= \frac{21x+2}{30} + \frac{1}{3} \\
7. \quad \frac{3x+7}{6} - \frac{5x-4}{3x-4} &= \frac{12x-17}{24} - \frac{1}{8} & 8. \quad \frac{12x+23}{18} - \frac{7x-2}{3x-10} &= \frac{8x-25}{12} + \frac{13}{36} \\
9. \quad \frac{10x-21}{14} - \frac{5x-4}{3x+2} &= \frac{5x-22}{7} + \frac{9}{14} & 10. \quad \frac{3x+2}{2x-5} - \frac{3(7-x)}{5} &= \frac{3x+4}{5} - \frac{1}{3} \\
11. \quad \frac{6x-7\frac{1}{2}}{13-2x} + 2x + \frac{2x}{3} &= 4\frac{3}{8} - \frac{25\frac{1}{4}-16x}{6} \\
12. \quad \frac{10x+19}{18} - \frac{3x+\frac{1}{2}}{\frac{13x}{4}-4} &= \frac{5x-3}{9}
\end{aligned}$$

Second Type :

Ex. 1 : Solve. $\frac{5}{x+2} + \frac{6}{x-4} = \frac{11}{x-1}$

Part 1 : $\frac{5}{x+2} + \frac{6}{x-4} = \frac{11}{x-1} \quad \therefore \frac{5(x-4) + 6(x+2)}{(x+2)(x-4)} = \frac{11}{x-1}$

$\therefore \frac{5x-20+6x+12}{x^2-2x-8} = \frac{11}{x-1} \quad \therefore \frac{11x-8}{x^2-2x-8} = \frac{11}{x-1}$

$\therefore (11x-8)(x-1) = 11(x^2-2x-8) \quad \therefore 11x^2-19x+8 = 11x^2-22x-88$

$\therefore 3x = -96 \quad \therefore x = -32$

Part 2 : Putting $x = -32$ in the given equation

$\frac{5}{(-32)+2} + \frac{6}{(-32)-4} \stackrel{?}{=} \frac{11}{(-32)-1} \quad \therefore \frac{5}{-30} + \frac{6}{-36} \stackrel{?}{=} \frac{11}{-33}$

$\therefore \left(-\frac{1}{6}\right) + \left(-\frac{1}{6}\right) \stackrel{?}{=} -\frac{1}{3} \quad \therefore -\frac{1}{3} \checkmark = -\frac{1}{3}$

$\therefore$ Solution set = $\{-32\}$.

Ex. 2 : Solve. $\frac{10}{5x-9} + \frac{14}{2x+9} = \frac{9}{x+8}$

Part 1 : $\frac{10}{5x-9} + \frac{14}{2x+9} = \frac{9}{x+8} \quad \therefore \frac{10(2x+9) + 14(5x-9)}{(5x-9)(2x+9)} = \frac{9}{x+8}$

$\therefore \frac{20x+90+70x-126}{10x^2+27x-81} = \frac{9}{x+8} \quad \therefore \frac{90x-36}{10x^2+27x-81} = \frac{9}{x+8}$

$\therefore \frac{9(10x-4)}{10x^2+27x-81} = \frac{9}{x+8} \quad \therefore \frac{10x-4}{10x^2+27x-81} = \frac{1}{x+8}$

$\therefore (10x-4)(x+8) = 10x^2+27x-81 \quad \therefore 10x^2+76x-32 = 10x^2+27x-81$

$\therefore 49x = -49 \quad \therefore x = -1.$

Part 2 : Putting $x = -1$ in the given equation

$$\frac{10}{5(-1)-9} + \frac{14}{2(-1)+9} \stackrel{?}{=} \frac{9}{(-1)+8} \quad \therefore \frac{10}{-14} + \frac{14}{7} \stackrel{?}{=} \frac{9}{7}$$

$$\therefore \frac{-5}{7} + \frac{14}{7} \stackrel{?}{=} \frac{9}{7} \quad \therefore \frac{9}{7} \stackrel{\checkmark}{=} \frac{9}{7}$$

$\therefore$ Solution set = $\{-1\}$.

EXERCISE 8.4

Solve.

$$1. \frac{1}{x+4} + \frac{1}{x+1} = \frac{2}{x-2}$$

$$2. \frac{2}{2x-5} + \frac{1}{x-5} = \frac{2}{x+5}$$

$$3. \frac{4}{x+2} + \frac{3}{x+1} = \frac{7}{x+3}$$

$$4. \frac{3}{x-6} + \frac{7}{x-2} = \frac{10}{x-4}$$

$$5. \frac{1}{x-2} + \frac{1}{x-3} = \frac{2}{x-1}$$

$$6. \frac{1}{x+1} + \frac{2}{x+2} = \frac{3}{x+3}$$

$$7. \frac{2}{x-3} - \frac{1}{x-7} = \frac{1}{x-5}$$

$$8. \frac{4}{x+2} - \frac{1}{x+3} = \frac{6}{2x+3}$$

$$9. \frac{7}{10x+1} = \frac{3}{2x+1} - \frac{4}{5x+1}$$

$$10. \frac{4}{4x+5} + \frac{3}{4x+1} = \frac{7}{4x+3}$$

Third Type :

Ex. 1 : Solve. $\frac{x+1}{x-1} + \frac{x+2}{x-2} = \frac{2x+8}{x+1}$

Part 1 : $\frac{x+1}{x-1} + \frac{x+2}{x-2} = \frac{2x+8}{x+1}$

$$\therefore \frac{(x-1)+2}{x-1} + \frac{(x-2)+4}{x-2} = \frac{(2x+2)+6}{x+1}$$

$$\therefore \frac{x-1}{x-1} + \frac{2}{x-1} + \frac{x-2}{x-2} + \frac{4}{x-2} = \frac{2(x+1)}{x+1} + \frac{6}{x+1}$$

$$\therefore 1 + \frac{2}{x-1} + 1 + \frac{4}{x-2} = 2 + \frac{6}{x+1} \quad \therefore \frac{2}{x-1} + \frac{4}{x-2} = \frac{6}{x+1}$$

$$\therefore \frac{2(x-2)+4(x-1)}{(x-1)(x-2)} = \frac{6}{x+1} \quad \therefore \frac{2x-4+4x-4}{x^2-3x+2} = \frac{6}{x+1}$$

$$\therefore \frac{6x-8}{x^2-3x+2} = \frac{6}{x+1}$$

$$\therefore (6x-8)(x+1) = 6(x^2-3x+2) \quad \therefore 6x^2-2x-8 = 6x^2-18x+12$$

$$\therefore 16x = 20 \quad \therefore x = \frac{5}{4}$$

Part 2 : Putting $x = \frac{5}{4}$ in the given equation—

$$\frac{(\frac{5}{4}) + 1}{(\frac{5}{4}) - 1} + \frac{(\frac{5}{4}) + 2}{(\frac{5}{4}) - 2} \stackrel{?}{=} \frac{2(\frac{5}{4}) + 8}{(\frac{5}{4}) + 1}$$

$$\frac{\frac{9}{4}}{\frac{1}{4}} + \frac{\frac{13}{4}}{-\frac{3}{4}} \stackrel{?}{=} \frac{\frac{42}{4}}{\frac{9}{4}}$$

$$9 - \frac{13}{3} \stackrel{?}{=} \frac{42}{9} \quad \therefore \frac{27 - 13}{3} \stackrel{?}{=} \frac{14}{3}$$

$$\frac{14}{3} \stackrel{\checkmark}{=} \frac{14}{3}$$

$$\therefore \text{Solution set} = \left\{ \frac{5}{4} \right\}$$

Ex. 2: Solve : $\frac{1}{x+1} + \frac{1}{x+2} = \frac{4x+7}{6x+6} - \frac{2x-1}{3x+6}$

Part 1: $\frac{1}{x+1} + \frac{1}{x+2} = \frac{4x+7}{6(x+1)} - \frac{2x-1}{3(x+2)}$

$$\therefore \frac{1}{x+1} - \frac{4x+7}{6(x+1)} = -\frac{1}{x+2} - \frac{2x-1}{3(x+2)}$$

$$\therefore \frac{6 - (4x+7)}{6(x+1)} = -\frac{3 + (2x-1)}{3(x+2)} \quad \therefore \frac{-4x-1}{6(x+1)} = \frac{-2x-2}{3(x+2)}$$

$$\therefore \frac{4x+1}{2(x+1)} = \frac{2x+2}{x+2} \quad \therefore \frac{(4x+4)-3}{2(x+1)} = \frac{(2x+4)-2}{x+2}$$

$$\therefore \frac{4(x+1)}{2(x+1)} - \frac{3}{2(x+1)} = \frac{2(x+2)}{x+2} - \frac{2}{x+2}$$

$$\therefore 2 - \frac{3}{2(x+1)} = 2 - \frac{2}{x+2} \quad \therefore -\frac{3}{2(x+1)} = -\frac{2}{x+2}$$

$$\therefore 3(x+2) = 4(x+1)$$

$$\therefore 3x+6 = 4x+4$$

$$\therefore x = 2$$

Part 2 : Putting $x = 2$ in the given equation

$$\frac{1}{(2)+1} + \frac{1}{(2)+2} \stackrel{?}{=} \frac{4(2)+7}{6(2)+6} - \frac{2(2)-1}{3(2)+6}$$

$$\frac{1}{3} + \frac{1}{4} \stackrel{?}{=} \frac{8+7}{12+6} - \frac{4-1}{6+6}$$

$$\frac{4+3}{12} \stackrel{?}{=} \frac{15}{18} - \frac{3}{12} \quad \therefore \frac{7}{12} \stackrel{?}{=} \frac{5}{6} - \frac{1}{4}$$

$$\frac{7}{12} \stackrel{?}{=} \frac{10-3}{12} \quad \therefore \frac{7}{12} \stackrel{\checkmark}{=} \frac{7}{12}$$

$$\therefore \text{Solution set} = \{ 2 \}.$$

EXERCISE 8.5

Solve.

$$1. \frac{2x+3}{x+1} = \frac{4x+5}{4x+4} + \frac{3x+3}{3x+1}$$

$$2. \frac{2x+1}{x-1} - \frac{2x-3}{2x-5} = \frac{3x+2}{3x-3}$$

$$3. \frac{2x+7}{6x-6} = \frac{x+1}{x-1} - \frac{2x+1}{3x-1}$$

$$4. \frac{x+5}{x+4} - \frac{2}{5x-1} = \frac{4x+17}{4x+16}$$

$$5. \frac{3x+5}{x+1} - \frac{10x+1}{6x+3} = \frac{4x+8}{3x+3}$$

$$6. \frac{x+4}{x+2} + \frac{x+5}{x+1} = \frac{2x+12}{x+3}$$

$$7. \frac{x+2}{2x+1} - \frac{3x+1}{9x-3} = \frac{4x+3}{8x+4} - \frac{2x-1}{6x-2}$$

$$8. \frac{5}{x+2} - \frac{4}{2x-3} = \frac{7}{12x-18} - \frac{6}{5x+10}$$

$$9. \frac{2(2x-5)}{x-3} + \frac{3x}{x-1} = \frac{7x-9}{x-2} \quad 10. \frac{2x+3}{2x-3} - \frac{x+2}{2-x} = 4\left(\frac{x+1}{2x-5}\right)$$

Fourth Type :

Ex. 1 : Solve. $\frac{1}{x-5} + \frac{1}{x-7} = \frac{1}{x-2} + \frac{1}{x-10}$

Part 1 : $\frac{1}{x-5} + \frac{1}{x-7} = \frac{1}{x-2} + \frac{1}{x-10}$

$$\therefore \frac{(x-7) + (x-5)}{(x-5)(x-7)} = \frac{(x-10) + (x-2)}{(x-2)(x-10)}$$

$$\therefore \frac{2x-12}{x^2-12x+35} = \frac{2x-12}{x^2-12x+20}$$

The numerator of each side of this equation is $2x-12$.

$\therefore$ Either $2x-12=0$ or $2x-12 \neq 0$. Considering both these possibilities, let us solve the equation.

Suppose $2x-12=0$

$$\therefore 2x=12 \quad \therefore x=6.$$

Now suppose $2x-12 \neq 0$, then the denominators of both sides of the equation must be equal.

$$\therefore x^2-12x+35 = x^2-12x+20$$

$$\therefore 35=20 \quad (\text{On cancelling } x^2-12x)$$

But this is absurd.

$$\therefore 2x-12=0 \text{ must hold. } \therefore x=6.$$

Part 2 : Putting $x=6$ in the given equation

$$\frac{1}{(6)-5} + \frac{1}{(6)-7} \stackrel{?}{=} \frac{1}{(6)-2} + \frac{1}{(6)-10}$$

$$\frac{1}{1} + \frac{1}{-1} \stackrel{?}{=} \frac{1}{4} + \frac{1}{-4} \quad \therefore 0 \stackrel{?}{=} 0$$

$$\therefore \text{Solution set} = \{ 6 \}$$

Another Method (For Part 1) :

$$\begin{aligned}\frac{1}{x-5} + \frac{1}{x-7} &= \frac{1}{x-2} + \frac{1}{x-10} \\ \therefore \frac{1}{x-5} - \frac{1}{x-2} &= \frac{1}{x-10} - \frac{1}{x-7} \\ \therefore \frac{(x-2) - (x-5)}{(x-5)(x-2)} &= \frac{(x-7) - (x-10)}{(x-10)(x-7)} \\ \therefore \frac{3}{x^2 - 7x + 10} &= \frac{3}{x^2 - 17x + 70} \\ \therefore x^2 - 7x + 10 &= x^2 - 17x + 70 \\ \therefore 10x &= 60 \quad \therefore x = 6\end{aligned}$$

Ex. 2 : Solve. $\frac{8}{x-3} + \frac{4}{x-9} = \frac{9}{x-4} + \frac{3}{x-8}$

Part 1 : $\frac{8}{x-3} + \frac{4}{x-9} = \frac{9}{x-4} + \frac{3}{x-8}$

$$\therefore \frac{8(x-9) + 4(x-3)}{(x-3)(x-9)} = \frac{9(x-8) + 3(x-4)}{(x-4)(x-8)}$$

$$\therefore \frac{8x - 72 + 4x - 12}{x^2 - 12x + 27} = \frac{9x - 72 + 3x - 12}{x^2 - 12x + 32}$$

$$\therefore \frac{12x - 84}{x^2 - 12x + 27} = \frac{12x - 84}{x^2 - 12x + 32}$$

The numerator of each side of this equation is $12x - 84$. Either $12x - 84 = 0$ or $12x - 84 \neq 0$. If $12x - 84 = 0$ then $12x = 84$.

$$\therefore x = 7$$

If $12x - 84 \neq 0$, then the denominators of both sides of the equation must be equal.

$$\therefore x^2 - 12x + 27 = x^2 - 12x + 32 \quad \therefore 27 = 32$$

But this is absurd.

$$\therefore 12x - 84 = 0 \text{ must hold.}$$

$$\therefore x = 7$$

Part 2: Putting $x = 7$ in the given equation

$$\frac{8}{(7)-3} + \frac{4}{(7)-9} \stackrel{?}{=} \frac{9}{(7)-4} + \frac{3}{(7)-8}$$

$$\frac{8}{4} + \frac{4}{-2} \stackrel{?}{=} \frac{9}{3} + \frac{3}{-1}$$

$$2 - 2 \stackrel{?}{=} 3 - 3$$

$$0 \checkmark 0$$

$$\therefore \text{Solution set} = \{7\}$$

Another Method (For Part 1) :

$$\frac{8}{x-3} + \frac{4}{x-9} = \frac{9}{x-4} + \frac{3}{x-8} \therefore \frac{8}{x-3} - \frac{9}{x-4} = \frac{3}{x-8} - \frac{4}{x-9}$$

$$\therefore \frac{8(x-4) - 9(x-3)}{(x-3)(x-4)} = \frac{3(x-9) - 4(x-8)}{(x-8)(x-9)}$$

$$\therefore \frac{8x - 32 - 9x + 27}{x^2 - 7x + 12} = \frac{3x - 27 - 4x + 32}{x^2 - 17x + 72}$$

$$\therefore \frac{-x - 5}{x^2 - 7x + 12} = \frac{-x + 5}{x^2 - 17x + 72} \therefore \frac{x + 5}{x^2 - 7x + 12} = \frac{x - 5}{x^2 - 17x + 72}$$

$$\therefore (x + 5)(x^2 - 17x + 72) = (x - 5)(x^2 - 7x + 12)$$

$$\therefore x^3 - 12x^2 - 13x + 360 = x^3 - 12x^2 + 47x - 60$$

$$\therefore -13x + 360 = 47x - 60$$

$$\therefore 420 = 60x \therefore x = 7.$$

EXERCISE 8-6

Solve.

$$1. \frac{1}{x-3} + \frac{1}{x-7} = \frac{1}{x-6} + \frac{1}{x-4} \quad 2. \frac{1}{x+5} + \frac{1}{x+7} = \frac{1}{x+10} + \frac{1}{x+2}$$

$$3. \frac{1}{x-2} - \frac{1}{x-4} = \frac{1}{x-6} - \frac{1}{x-8} \quad 4. \frac{2}{2x+5} + \frac{1}{x-1} = \frac{2}{2x-3} + \frac{1}{x+3}$$

$$5. \frac{1}{2x-1} + \frac{8}{4x-1} = \frac{6}{3x-1} + \frac{3}{6x-1} \quad 6. \frac{21}{x+7} - \frac{16}{x+4} = \frac{21}{x-9} - \frac{16}{x-17}$$

$$7. \frac{5}{x-6} - \frac{3}{2-x} = \frac{6}{x-5} - \frac{2}{3-x} \quad 8. \frac{4}{x-9} - \frac{8}{3-x} = \frac{9}{x-4} - \frac{3}{8-x}$$

$$9. \frac{6}{x+5} + \frac{2}{1-x} = \frac{5}{x+6} - \frac{1}{x-2} \quad 10. \frac{10}{2x+1} - \frac{2}{2x-3} = \frac{15}{3x+2} - \frac{3}{3x-2}$$

Fifth Type :

$$\text{Ex. 1: Solve. } \frac{x+2}{x+1} + \frac{x+5}{x+4} = \frac{x+3}{x+2} + \frac{x+4}{x+3}$$

$$\text{Part 1: } \frac{x+2}{x+1} + \frac{x+5}{x+4} = \frac{x+3}{x+2} + \frac{x+4}{x+3}$$

$$\therefore \frac{(x+1)+1}{x+1} + \frac{(x+4)+1}{x+4} = \frac{(x+2)+1}{x+2} + \frac{(x+3)+1}{x+3}$$

$$\therefore \frac{x+1}{x+1} + \frac{1}{x+4} + \frac{x+4}{x+4} + \frac{1}{x+3}$$

$$= \frac{x+2}{x+2} + \frac{1}{x+2} + \frac{x+3}{x+3} + \frac{1}{x+3}$$

$$\therefore 1 + \frac{1}{x+1} + 1 + \frac{1}{x+4} = 1 + \frac{1}{x+2} + 1 + \frac{1}{x+3}$$

$$\therefore \frac{1}{x+1} + \frac{1}{x+4} = \frac{1}{x+2} + \frac{1}{x+3}$$

$$\therefore \frac{(x+4) + (x+1)}{(x+1)(x+4)} = \frac{(x+3) + (x+2)}{(x+2)(x+3)}$$

$$\therefore \frac{2x+5}{x^2+5x+4} = \frac{2x+5}{x^2+5x+6}$$

$$\therefore (2x+5)(x^2+5x+6) = (2x+5)(x^2+5x+4)$$

$$\therefore (2x+5)(x^2+5x+6) - (2x+5)(x^2+5x+4) = 0$$

$$\therefore (2x+5)(x^2+5x+6 - x^2 - 5x - 4) = 0$$

$$\therefore (2x+5)(2) = 0$$

$$\therefore 2 \neq 0 \therefore 2x+5=0 \therefore 2x=-5 \therefore x=-\frac{5}{2}$$

Part 2: Putting $x = -\frac{5}{2}$ in the given equation

$$\frac{(-\frac{5}{2})+2}{(-\frac{5}{2})+1} + \frac{(-\frac{5}{2})+5}{(-\frac{5}{2})+4} \stackrel{?}{=} \frac{(-\frac{5}{2})+3}{(-\frac{5}{2})+2} + \frac{(-\frac{5}{2})+4}{(-\frac{5}{2})+3}$$

$$\frac{-\frac{1}{2}}{-\frac{3}{2}} + \frac{\frac{5}{2}}{\frac{3}{2}} \stackrel{?}{=} \frac{-\frac{1}{2}}{-\frac{1}{2}} + \frac{\frac{3}{2}}{\frac{1}{2}} \therefore \frac{1}{3} + \frac{5}{3} \stackrel{?}{=} -1 + 3$$

$$\frac{6}{3} \stackrel{?}{=} 2 \therefore 2 \checkmark 2$$

$$\therefore \text{Solution Set} = \{-\frac{5}{2}\}$$

Another Method (For Part 1):

$$\frac{x+2}{x+1} + \frac{x+5}{x+4} = \frac{x+3}{x+2} + \frac{x+4}{x+3}$$

$$\therefore \frac{(x+1)+1}{x+1} + \frac{(x+4)+1}{x+4} = \frac{(x+2)+1}{x+2} + \frac{(x+3)+1}{x+3}$$

$$\therefore 1 + \frac{1}{x+1} + 1 + \frac{1}{x+4} = 1 + \frac{1}{x+2} + 1 + \frac{1}{x+3}$$

$$\therefore \frac{1}{x+1} + \frac{1}{x+4} = \frac{1}{x+2} + \frac{1}{x+3}$$

$$\therefore \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{x+3} - \frac{1}{x+4}$$

$$\therefore \frac{(x+2) - (x+1)}{(x+1)(x+2)} = \frac{(x+4) - (x+3)}{(x+3)(x+4)}$$

$$\therefore \frac{1}{x^2+3x+2} = \frac{1}{x^2+7x+12}$$

$$\therefore x^2+7x+12 = x^2+3x+2$$

$$\therefore 4x = -10 \therefore x = -\frac{5}{2}$$

Ex. 2: Solve. $\frac{2x+11}{x+5} - \frac{9x-9}{3x-4} = \frac{4x+13}{x+3} - \frac{15x-47}{3x-10}$

Part 1: $\frac{2x+11}{x+5} - \frac{9x-9}{3x-4} = \frac{4x+13}{x+3} - \frac{15x-47}{3x-10}$

$$\therefore \frac{(2x+10)+1}{x+5} - \frac{(9x-12)+3}{3x-4} = \frac{(4x+12)+1}{x+3} - \frac{(15x-50)+3}{3x-10}$$

$$\therefore \frac{2(x+5)}{x+5} + \frac{1}{x+5} - \left[\frac{3(3x-4)}{3x-4} + \frac{3}{3x-4} \right]$$

$$= \frac{4(x+3)}{x+3} + \frac{1}{x+3} - \left[\frac{5(3x-10)}{3x-10} + \frac{3}{3x-10} \right]$$

$$\therefore 2 + \frac{1}{x+5} - 3 - \frac{3}{3x-4} = 4 + \frac{1}{x+3} - 5 - \frac{3}{3x-10}$$

$$\therefore \frac{1}{x+5} - \frac{3}{3x-4} = \frac{1}{x+3} - \frac{3}{3x-10}$$

$$\therefore \frac{(3x-4) - 3(x+5)}{(x+5)(3x-4)} = \frac{(3x-10) - 3(x+3)}{(x+3)(3x-10)}$$

$$\therefore \frac{3x-4-3x-15}{3x^2+11x-20} = \frac{3x-10-3x-9}{3x^2-x-30}$$

$$\therefore \frac{-19}{3x^2+11x-20} = \frac{-19}{3x^2-x-30}$$

$$\therefore 3x^2+11x-20 = 3x^2-x-30$$

$$\therefore 12x = -10 \therefore x = -\frac{5}{6}$$

Part 2: Putting $x = -\frac{5}{6}$ in the given equation

$$\frac{2(-\frac{5}{6})+11}{(-\frac{5}{6})+5} - \frac{9(-\frac{5}{6})-9}{3(-\frac{5}{6})-4} \stackrel{?}{=} \frac{4(-\frac{5}{6})+13}{(-\frac{5}{6})+3} - \frac{15(-\frac{5}{6})-47}{3(-\frac{5}{6})-10}$$

$$\frac{-\frac{5}{3}+11}{-\frac{5}{6}+5} - \frac{-\frac{15}{2}-9}{-\frac{5}{2}-4} \stackrel{?}{=} \frac{-\frac{10}{3}+13}{-\frac{5}{6}+3} - \frac{-\frac{25}{2}-47}{-\frac{5}{2}-10}$$

$$\frac{\frac{28}{3}}{\frac{26}{6}} - \frac{-\frac{33}{2}}{-\frac{13}{2}} \stackrel{?}{=} \frac{\frac{29}{3}}{\frac{13}{6}} - \frac{-\frac{119}{2}}{-\frac{25}{2}}$$

$$\frac{28}{3} \times \frac{6}{26} - \frac{33}{2} \times \frac{2}{13} \stackrel{?}{=} \frac{29}{3} \times \frac{6}{13} - \frac{119}{2} \times \frac{2}{25}$$

$$\frac{56}{25} - \frac{33}{13} \stackrel{?}{=} \frac{58}{13} - \frac{119}{25}$$

$$\frac{728-825}{325} \stackrel{?}{=} \frac{1450-1547}{325}$$

$$\frac{-97}{325} \stackrel{\checkmark}{=} \frac{-97}{325}$$

$$\therefore \text{Solution Set} = \left\{ -\frac{5}{6} \right\}$$

EXERCISE 8.7

Solve.

$$1. \frac{x+3}{x+2} + \frac{x+11}{x+10} = \frac{x+5}{x+4} + \frac{x+9}{x+8} \quad 2. \frac{x+4}{x+3} + \frac{x+5}{x+4} = \frac{x+3}{x+2} + \frac{x+6}{x+5}$$

$$3. \frac{x}{x-2} - \frac{x-8}{x-6} = \frac{x+1}{x-1} - \frac{x-9}{x-7} \quad 4. \frac{x-6}{x-7} - \frac{x-9}{x-10} = \frac{x-7}{x-8} - \frac{x-10}{x-11}$$

$$5. \frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{x-5}{x-6} - \frac{x-6}{x-7} \quad 6. \frac{x-7}{x-9} + \frac{x-3}{x-5} = \frac{x-4}{x-6} + \frac{x-6}{x-8}$$

$$7. \frac{2x+9}{x+4} + \frac{3x-2}{x-1} = \frac{4x+13}{x+3} + \frac{x+1}{x}$$

$$8. \frac{5x-34}{x-7} + \frac{3x-26}{x-9} = \frac{5x-24}{x-5} + \frac{3x-32}{x-11}$$

$$9. \frac{4x-17}{x-4} + \frac{10x-13}{2x-3} = \frac{8x-30}{2x-7} - \frac{5x-4}{1-x}$$

$$10. \frac{4x-15}{x-4} + \frac{5x-39}{x-8} = \frac{4x-3}{x-1} + \frac{5x-54}{x-11}$$

$$11. \frac{5x-19}{x-4} + \frac{8x-71}{x-9} = \frac{6x-41}{x-7} + \frac{7x-41}{x-6}$$

$$12. \frac{8x-7}{x-1} + \frac{7x-48}{x-7} = \frac{11x-32}{x-3} + \frac{4x-19}{x-5}$$

$$13. \frac{6x-10}{x-2} - \frac{8-x}{6-x} = \frac{6x-4}{x-1} + \frac{x-9}{7-x}$$

$$14. \frac{4x+17}{x+4} - \frac{5x+36}{x+7} = \frac{2x+7}{x+3} - \frac{3x+19}{x+6}$$

$$15. \frac{7x-20}{x-3} + \frac{13-3x}{4-x} - \frac{2x-15}{x-7} = \frac{8x-47}{x-6}$$

Chapter 9 : GRAPHS — 2

§ 9.1 Graphs of Quadratic Equations :

The equations of the form, $y = mx + h$ or $Ax + By + C = 0$ are called equations of the first degree in two variables. Their graphs are straight lines. We shall now concern ourselves with the graphs of equations containing the second power of x ; i.e. equations of the type $y = x^2$ or $y = ax^2$ or $y = ax^2 + bx + c$.

Let us consider the simplest of these equations, namely, $y = x^2$. Let us tabulate some corresponding values of x and y . Complete the following table—

$$y = x^2$$

x	0	1	2	3	4	-1	-2	-3	-4
$y = x^2$	0		4					9	

Plot the above points on a co-ordinate system and note the following points.

(1) Observe whether the values of y are positive, negative or zero. From your observation decide the quadrants in which the graph will lie and choose your axes in such a way that a large part of your graph-paper may be used. Let the X-axis be near the lower edge of the graph-paper (See fig. 9.1).

(2) Find out from the table whether the absolute values of x which give the same value of y are equal. If there are some such values what important aspect about the graph do they bring out?

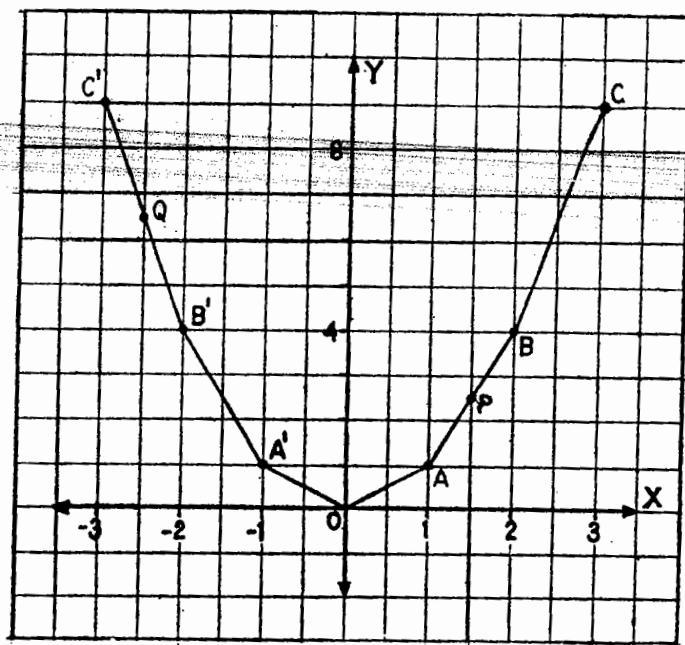


Fig. 9.1

(3) Do you feel that the points plotted would be on one straight line? If not, draw the segments joining consecutive points plotted by you.

Let us verify whether the open polygon obtained in the above figure is the graph of $y = x^2$. Let us take the mid-point P of the segment AB. Its co-ordinates are (1.5, 2.5). If the above polygon is the required graph, then the co-ordinates of P must satisfy the equation $y = x^2$. Let us see.

$$\text{Is } 2.5 \stackrel{?}{=} (1.5)^2 ?$$

$$\text{That is, is } 2.5 \stackrel{?}{=} 2.25 ? \text{ No !}$$

∴ The equation $y = x^2$ is not satisfied by the point $(1.5, 2.5)$. That is, P does not lie on the expected graph.

We may consider another point Q on the said open polygon, whose co-ordinates are $(-2.5, 6.5)$. Do these co-ordinates satisfy the equation $y = x^2$?

$$\text{Is } 6.5 \stackrel{?}{=} (2.5)^2 ?$$

That is, is $6.5 \stackrel{?}{=} 6.25$? No !

This means that the point Q on the segment B' C' also does not lie on the required graph. So we can certainly say that the figure obtained by us by plotting a few points above, is not the graph of $y = x^2$. The graph of $y = x^2$ is a curve. Its curvedness is changing continually. In order to draw an accurate graph we must enrich our table by taking more values of x between each pair of two consecutive values already taken. As we plot points nearer and nearer to each other, our graph would be more and more accurate.

Now complete the following table, plot the points on a new co-ordinate system and try to draw a continuous smooth curve.

$$y = x^2$$

x	0	0.5	-0.5	1	-1	1.5	-1.5	2	-2	2.5	-2.5	3	-3
$y = x^2$													

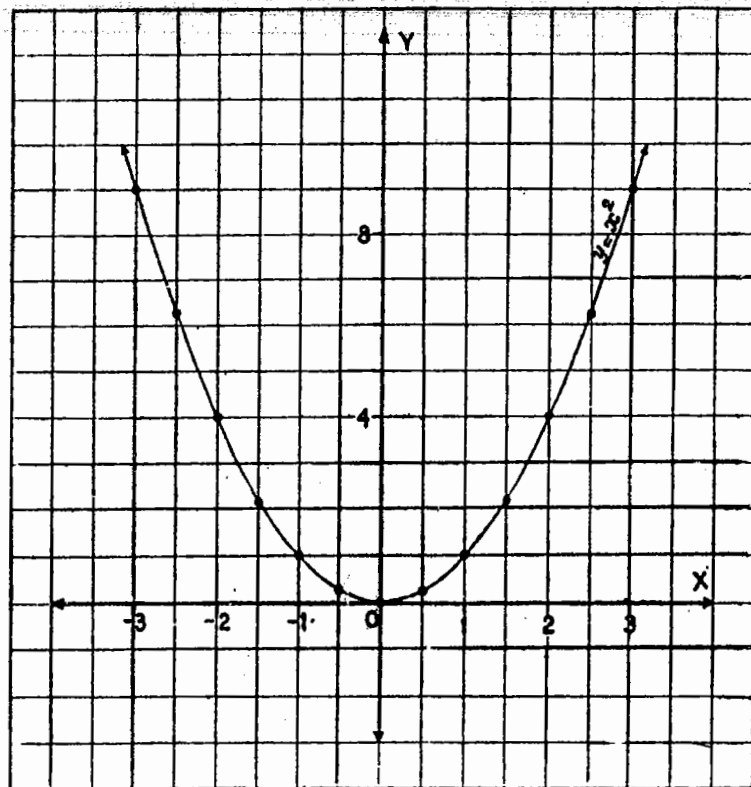


Fig. 9.2

Observe the shape of the curve in the above figure. Such curves are called **parabolas**. The point like $(0, 0)$ is called the **vertex** of the parabola.

§ 9.2 Graph of $y = -x^2$:

See fig. 9.3. You will observe that the graph of $y = -x^2$ is the reflection of the graph of $y = x^2$ in the X-axis.

§ 9.3 Graphs of $y = x^2 + h$ and $y = -x^2 + h$:

We have seen that the graph of $y = x + h$ is obtained by translating the graph of $y = x$, through the distance $|h|$ above the X-axis if $h > 0$ and below it if $h < 0$.

Every point (x, y) on the line $y = x$ is translated to $(x, y + h)$. We have noticed the same effect in case of the lines $y = mx$ and $y = mx + h$. Every point (x, y) on the line $y = mx$ is translated to $(x, y + h)$ to the line $y = mx + h$.

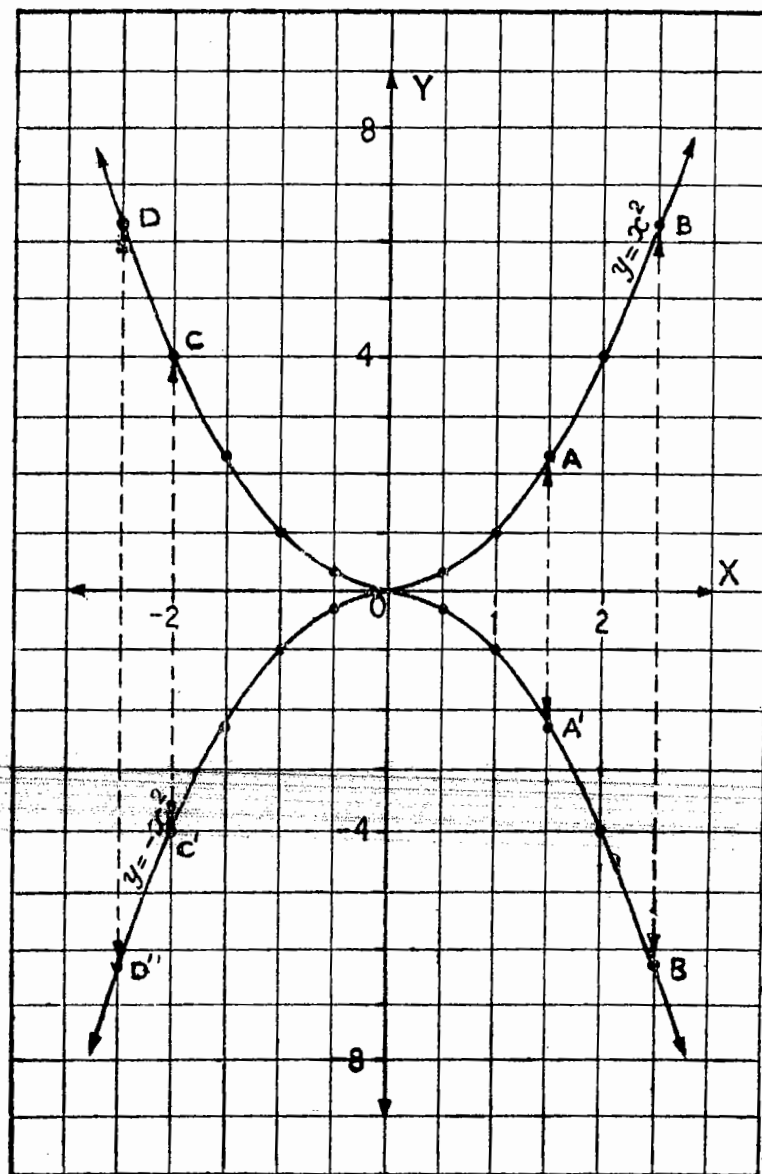


Fig. 9.3

This same principle holds in case of the graphs of $y = x^2$ and $y = x^2 + h$. If every point on $y = x^2$ is translated through ' h ' in a direction perpendicular to the X-axis, we get the curve $y = x^2 + h$. Let us verify this fact by drawing the two graphs of $y = x^2$ and $y = x^2 + 3$ on the same co-ordinate system. Prepare the following tables.

$$y = x^2$$

x	0	0.5	1	-0.5	-1	1.5	2	-1.5	-2
$y = x^2$									

$$y = x^2 + 3$$

x	0	0.5	1	-0.5	-1	1.5	2	-1.5	-2
$y = x^2 + 3$	3			3.25			7		

The graphs obtained by plotting the points in the above tables are shown in fig. 9.4 below, which also includes the graph of $y = x^2 - 4$.

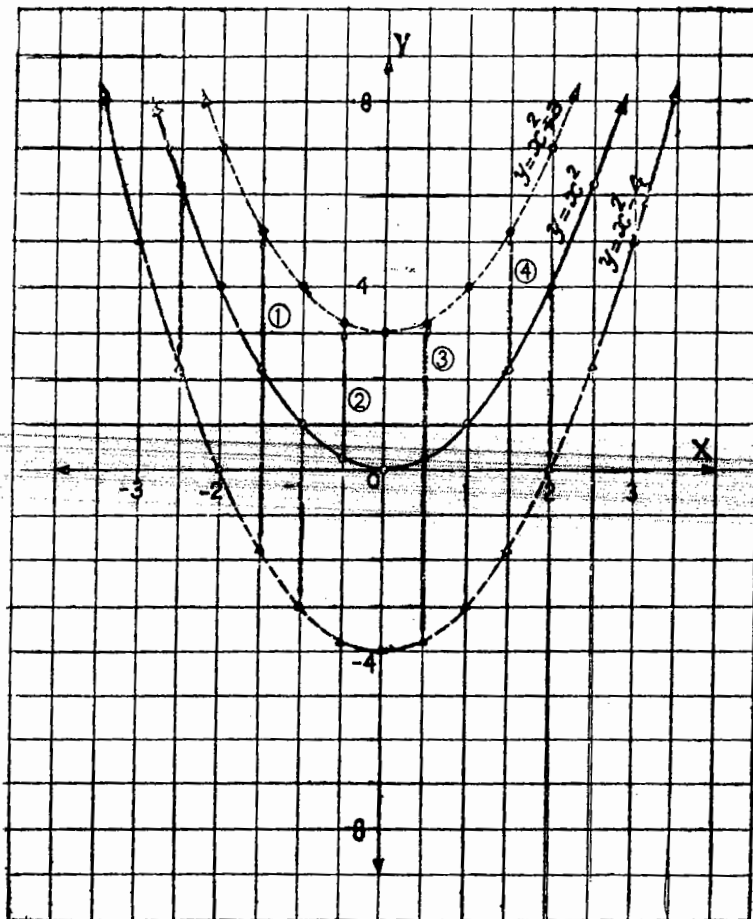


Fig. 9.4

Observe from the tables as well as from the graphs the relation between the points on $y = x^2$ and $y = x^2 + 3$ for which the values of x are the same.

The graph of $y = x^2 - 4$ can be obtained by translating every point on the curve $y = x^2$, (-4) units along a direction parallel to the Y-axis.

The graphs of $y = x^2 + 3$ and $y = x^2 - 4$, are also parabolas and they are symmetrical with respect to the Y-axis. What are the vertices of the curves $y = x^2 + 3$ and $y = x^2 - 4$?

Now, find the answer to the following question carefully. You can see that the gap between any two curves of $y = x^2$, $y = x^2 + 3$ and $y = x^2 - 4$ is very wide at $x = 0$, as compared to the gaps at $x = 1$ or $x = 2$ or $x = -1$ or $x = -2$. Do you think that these curves would meet at some point?

It may be stated, that these graphs never meet each other however narrow the horizontal gap between them may become.

From the above discussion, the relation between the graphs of $y = -x^2$, $y = -x^2 + 3$ and $y = -x^2 - 4$ can well be imagined. On the same co-ordinate

system draw the graphs of $y = -x^2$, $y = -x^2 + 3$ and $y = -x^2 - 4$ with the help of the following tables.

$$y = -x^2$$

x	0	0.5	-0.5	1	-1	1.5	-1.5	2	-2
$y = -x^2$									

$$y = -x^2 + 3$$

x	0	0.5	-0.5	1	-1	1.5	-1.5	2	-2
$y = -x^2 + 3$	3		2.75			0.75			

$$y = -x^2 - 4$$

x	0	0.5	-0.5	1	-1	1.5	-1.5	2	-2
$y = -x^2 - 4$			-4.25					-8	

The graphs obtained from the above tables are shown in fig. 9.5 below.

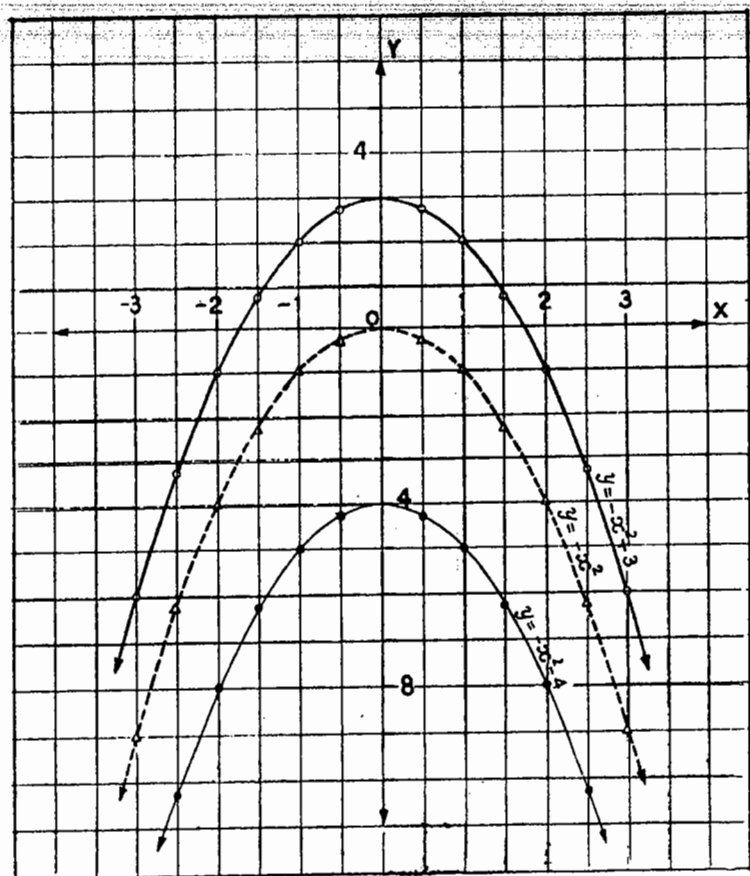


Fig. 9.5

EXERCISE 9.1

Draw the graphs of the following equations on one and the same co-ordinate system. Let the values of x be such that $-3 \leq x \leq 3$.

1. $y = x^2$
2. $y = x^2 - 3$
3. $y = x^2 + 4$

Draw the graphs of the following equations on one and the same co-ordinate system. Let the values of x be such that $-4 \leq x \leq 4$.

4. $y = -x^2$
5. $y = 2 - x^2$
6. $y = -x^2 - 3$

7. Draw the graph of $y = x^2$. Use the method of translation for some of the points on the graph to obtain the graph of $y = x^2 - 4$.

8. Draw the graph of $y = -x^2$. Translate some of the points on this graph and obtain the graph of $y = 4 - x^2$.

§ 9.4 Graph of $y = ax^2$:

We have seen how the graph of $y = x^2$ or $y = -x^2$ is displaced by adding a real number to the right hand side of the equation $y = x^2$ or $y = -x^2$. Let us now see the effect of multiplying the term x^2 by a real number.

Prepare the following table for drawing the graph of $y = 2x^2$.

$$y = 2x^2$$

x	0	0.5	-0.5	1	-1	1.5	-1.5	2	-2
$y = 2x^2$									

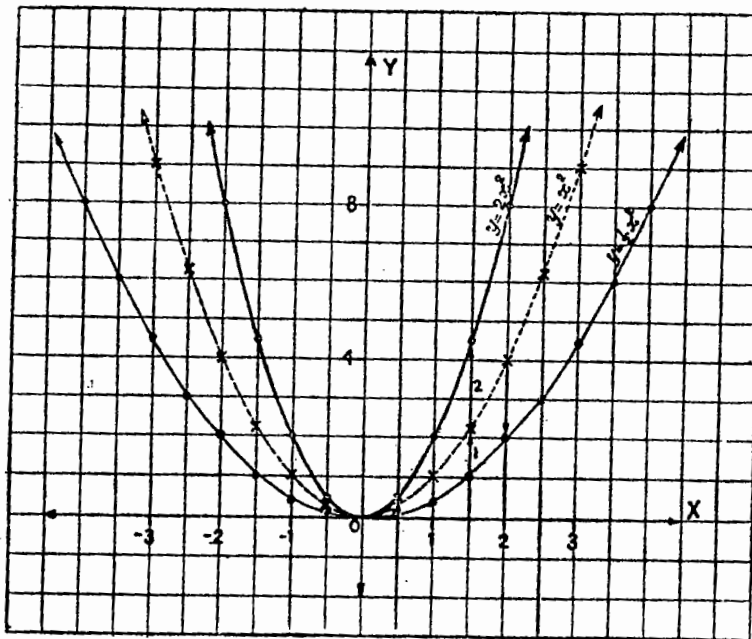


Fig. 9.6

Draw the graph of $y = x^2$ on the same co-ordinate system and compare the two graphs. The ordinate of every point on the graph of $y = x^2$ is doubled in the case of the graph of $y = 2x^2$. So, for the same abscissa, the ordinate being double, the graph of $y = 2x^2$ becomes narrow. If you draw the graph of $y = \frac{1}{2}x^2$ on the same co-

ordinate system, you will see that the ordinate of every point on $y = x^2$ is halved on the graph of $y = \frac{1}{2}x^2$, the result being that of broadening the span between the two parts of the parabola. See fig. 9.6 above.

From the above graphs it may be concluded that the graph of $y = ax^2$, ($a > 0$), is a parabola; the span of the graph along the horizontal (or the X-axis) becomes shorter as the value of 'a' (the coefficient of x^2) increases and the span becomes wider as the value of 'a' decreases.

All the graphs of the equations of the form $y = ax^2$, ($a > 0$), pass through the origin, which is the vertex of all those parabolas. The Y-axis is the axis of symmetry for each of them.

EXERCISE 9.2

On the same co-ordinate system draw the graphs of—

1. $y = -x^2$
2. $y = -\frac{1}{2}x^2$
3. $y = -2x^2$
4. $y = -3x^2$

Observe how the span of the graph changes with the coefficient of x^2 . State what you observe.

5. Draw the graph of $y = -2x^2$. Translate every point you have plotted 3 units in a direction parallel to the Y-axis and draw the curve passing through the images. What is the equation of the new curve?
6. Draw the graph of $y = 3x^2$. State how you would obtain the graph of $y = 3x^2 - 6$ from it by translation. Translate the points you have plotted for the curve $y = 3x^2$ and then draw the curve representing $y = 3x^2 - 6$ [Note : Choose a small unit on the Y-axis.]

§ 9.5 Graph of $y = ax^2 + h$:

From the graphs drawn by you for examples 5 and 6 in the above exercise, you must have noticed the effect of a constant term added to the right hand side of the equation $y = ax^2$.

Increase in the absolute value of the coefficient of x^2 in $y = \pm ax^2$, ($a \neq 0$), results in making the span shorter; the effect of adding a constant term is to displace the graph in a direction parallel to the Y-axis.

The vertex of the parabola $y = ax^2$, ($a \in \mathbb{R}$, $a \neq 0$) is (0, 0), while the vertex of the parabola $y = ax^2 + h$ is shifted to (0, h). The curves of the equations of the type $y = ax^2 + h$, ($a \neq 0$), are all parabolas, which are symmetrical about the Y-axis.

EXERCISE 9.3

State the co-ordinates of the vertices of the following curves.

1. $y = -x^2$;
2. $y + 2 = x^2$;
3. $y = 2 - x^2$;
4. $2x^2 + y = 3$;
5. $3 - 4x^2 = y$.
6. Say whether the point (3, -18) lies on the graph of $y = -2x^2$. If it is on the graph what are the co-ordinates of the point symmetrically placed on the graph with reference to the Y-axis?
7. Draw the graph of $y = x^2$. Draw the line $y = 6$ on the same co-ordinate system. State the co-ordinates of the points of intersection of the two graphs. Can you say why the line $y = -8$ does not intersect the graph of $y = x^2$?
8. Read off the approximate values (correct to first decimal place) of the x-co-ordinates of the points on the curve $y = x^2$ whose y-co-ordinates are 4, 7, 10 respectively.

§ 9.6 Square Root :

In your course in Algebra for Std. IX, you have noticed that the operation of taking square roots is meaningful only for non-negative real numbers. Sometimes, the square roots of those real numbers which are not perfect squares of rational numbers have to be reckoned approximately upto a required degree of accuracy. We can employ the graph of $y = x^2$ in estimating the square roots of positive real numbers. Let us see how.

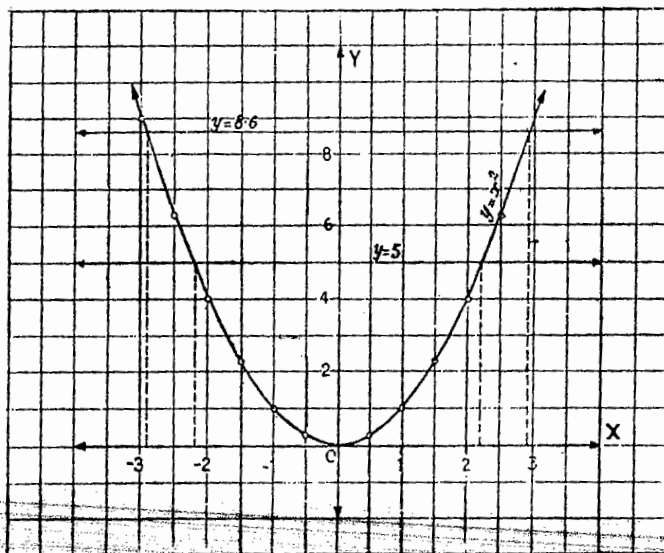


Fig. 9.7 shows the graph of $y = x^2$ and $y = 5$. The two graphs intersect in two points $(-2.2, 5)$ and $(2.2, 5)$; the x -co-ordinates of the points are stated approximately.

$y = 5$ and $y = x^2$ can be combined into one equation as $x^2 = 5$. So, when $y = 5$, that is when $x^2 = 5$, the value of the x -co-ordinate naturally gives the square root of 5. The answer is readily supplied by the x -

Fig. 9.7

ordinates of the points of intersection of the two graphs.

$$\therefore \sqrt{5} \approx 2.2 \text{ and } -\sqrt{5} \approx -2.2$$

In the same way, to find the approximate square roots of 8.6, we draw the line $y = 8.6$ and read the x -co-ordinates of the points of intersection of the graphs of $y = 8.6$ and $y = x^2$.

Thus we have, $\sqrt{8.6} \approx 2.9$ and $-\sqrt{8.6} \approx -2.9$.

EXERCISE 9.4

Establish a co-ordinate system such that you can read off values of the co-ordinates of points correct to one place of decimals. On this co-ordinate system draw the graph of $y = x^2$ for values of x such that $-3 \leq x \leq 3$. Use this graph to read off the values of the following.

- | | | | |
|-----------------|-----------------|------------------|------------------|
| 1. $\sqrt{6}$ | 2. $-\sqrt{6}$ | 3. $-\sqrt{8}$ | 4. $-\sqrt{3}$ |
| 5. $\sqrt{7.4}$ | 6. $\sqrt{8.2}$ | 7. $-\sqrt{4.8}$ | 8. $-\sqrt{1.9}$ |

§ 9.7 Graph of $y = (x - h)^2$:

We have noticed the effect of ' a ' and ' h ' in the graph of $y = ax^2 + h$. Now let us investigate what change is caused in the graph of $y = x^2$ by taking $(x - h)$ instead of x . Let us take a simple example. We try to draw the graph of $y = (x - 2)^2$. We tabulate some of the values of x and y .

$$y = (x - 2)^2$$

x	0	1	-1	2	3	4	5	0.5	-0.5	1.5	2.5	3.5
$y = (x - 2)^2$	4	1	9	0	1	4	9	2.25	6.25	0.25	0.25	2.25

From the above table, you can find points which are symmetrical, i.e. whose y -co-ordinates are equal. The point $(2, 0)$ is the only point which is its own partner. So, it must be the vertex of the parabola. Thus, the effect of (-2) in $(x - 2)^2$ is in shifting the vertex of the parabola $y = x^2$ to $(2, 0)$, and the axis of symmetry becomes

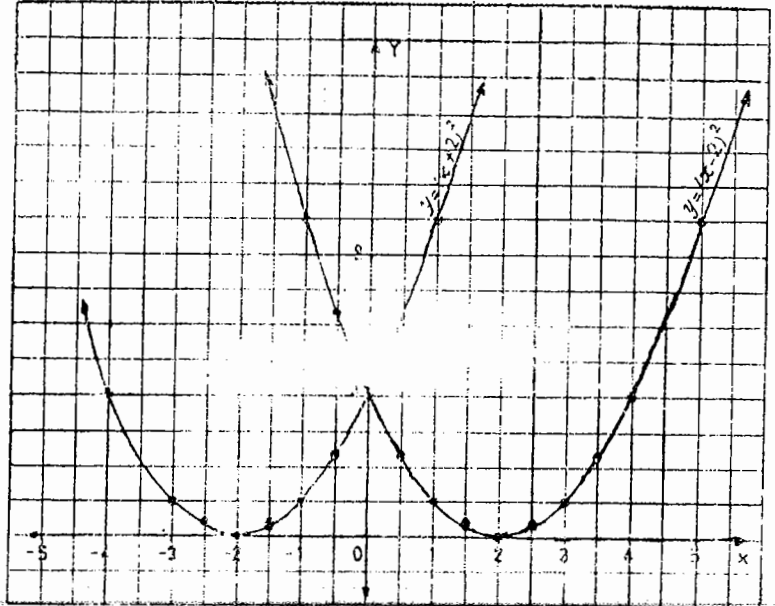


Fig. 9.8

the line $x = 2$. See fig. 9.8 above. In the same figure, the graph of $y = (x + 2)^2$ is also shown. You can notice the effect of the term $(+2)$ in $(x + 2)^2$. What is it?

EXERCISE 9.5

- Express the equation $y = x^2 - 6x + 9$ in a suitable form and state the vertex and the line of symmetry of its graph.
- What is the vertex and what is the line of symmetry of the parabola $y = x^2 + 4x + 4$?
- Can you guess how the graph of $y = 4x - 4 - x^2$ would be? What would be the vertex and the line of symmetry?
- Draw the graphs of the equations in examples 1 to 3 above on the same co-ordinate system.

§ 9.8 Graph of $y = ax^2 + bx + c$:

Let us consider the equation $y = x^2 - 2x - 3$. We can transform this equation in a suitable form and write it as $y = (x - 1)^2 - 4$, which is of the form $y = X^2 - 4$, where $X = x - 1$.

We can now interpret easily the form of the graph. The effect of -4 in $y = X^2 - 4$ would be to shift the vertex of $y = X^2$ to $(0, -4)$, while $y = X^2$ has its vertex at $(1, 0)$. So, the vertex of the required graph is at $(1, -4)$.

Observe the figure 9.9 on the next page and see how the graph of $y = x^2$ is shifted twice to obtain the graph of $y = (x - 1)^2 - 4$, i.e., of $y = x^2 - 2x - 3$.

We can draw the graph of any equation of the form $y = x^2 + bx + c$ by writing it as a sum of two terms — one a square term of the form $(x + h)^2$ and second a constant term. Let us consider the equation

$$y = x^2 + 4x + 3$$

$$= x^2 + 4x + 4 - 1$$

$$= (x + 2)^2 - 1.$$

Now, we tabulate some of the values of x and y . Noting from the above equation that the vertex is at the point $(-2, -1)$.

See fig. 9.10.

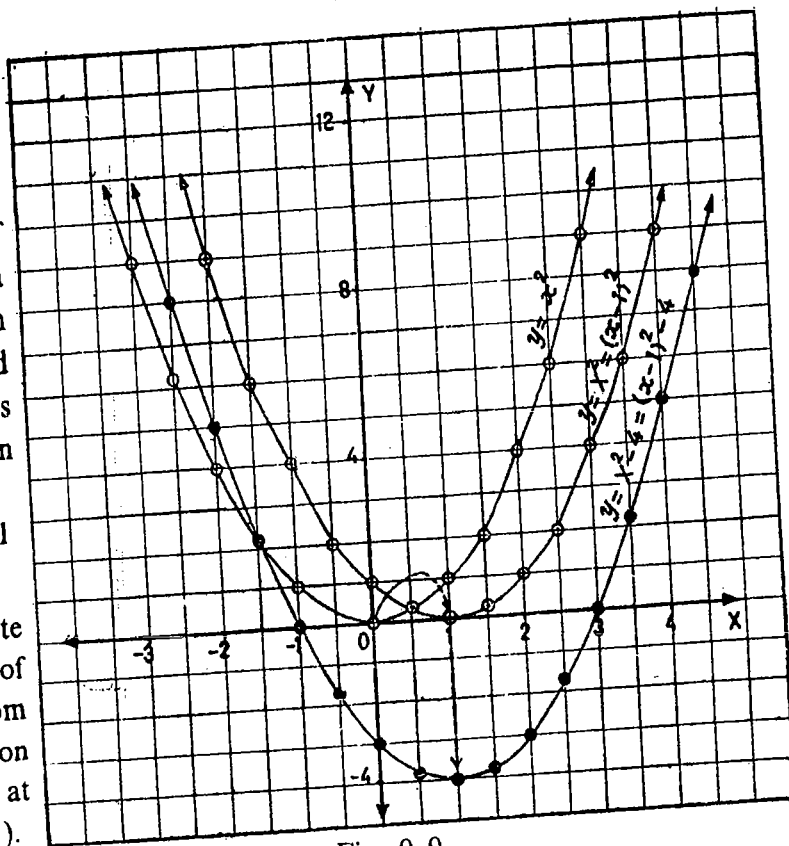


Fig. 9.9

$$y = x^2 + 4x + 3$$

x	-2	-1	-3	0	-4	1	-5	2	-6
$y = (x + 2)^2 - 1$	-1	0	0	3	3	8	8	15	15

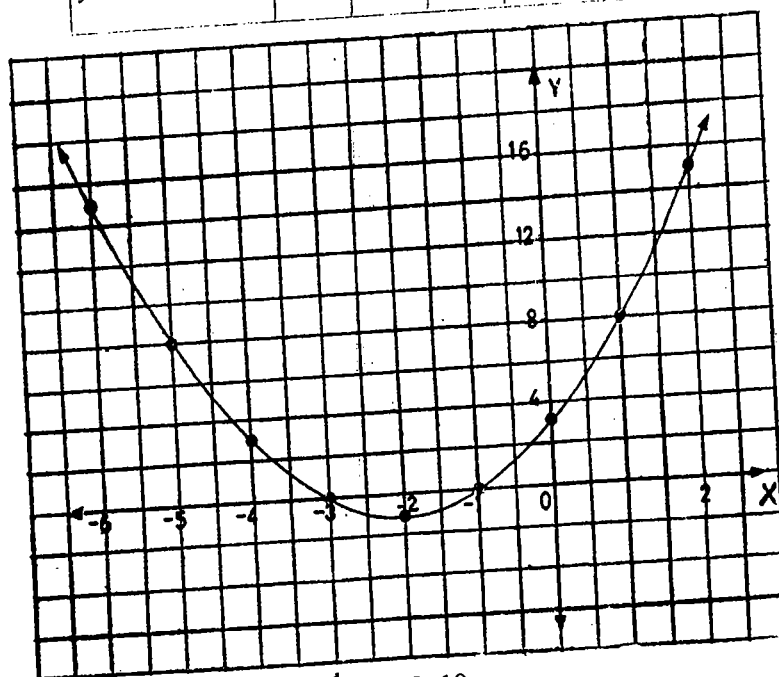


Fig. 9.10

Thus we see that it is easy to draw the graph of the equation $y = x^2 + bx + c$, once we put it in a form such that in the r. h. s. its form can be expressed as the sum of two terms — one a complete square, and another a constant term. The constant in the square-term and the remaining constant term determine the vertex of the parabola.

The equation $y = x^2 - 2x - 3$ can be put in the form $y = (x - 1)^2 - 4$ and the vertex turns out to be $(1, -4)$. In the case of the equation $y = x^2 + 4x + 3$ it is transformed into $y = (x + 2)^2 - 1$ giving the vertex at $(-2, -1)$. Similarly, we can put the equation $y = x^2 + 6x + 14$ in the form $y = (x + 3)^2 + 5$ giving the vertex at $(-3, 5)$.

The discussion of the graphs of the general quadratic equation of the form $y = ax^2 + bx + c$ in which the coefficient of x^2 is other than unity (i.e. ± 1), is not done here.

EXERCISE 9.6

1. Write the equation $y = x^2 - 6x + 9$ in a complete-square form and determine the vertex and the line of symmetry of its graph. Make a table of values of x and y , for four values of x each greater than and less than the x -co-ordinate of the vertex, and draw the graph.

Proceed as follows : $y = x^2 - 6x + 9 = (x - 3)^2$. $\therefore (3, 0)$ is the vertex and $x = 3$ is the line of symmetry. The table of values is as follows.

x	3	2	4	2.5	3.5	1.5	4.5	1	5
$y = (x - 3)^2$									

2. What are the co-ordinates of the vertex of the parabola $y = x^2 + 4x + 4$? What is the line of symmetry? Prepare a table of values of x and y for values of x such that $-4 \leq x \leq 0$ and draw the graph.
3. In which quadrants does the graph of $y = 4x - 4 - x^2$ lie? What is its vertex and what is the axis of symmetry? Draw the graph.
4. Write the equation $y = x^2 - 2x - 5$ as $y = (x - 1)^2 - 6$. Determine the vertex and the axis of symmetry. Draw the graph of the equation.
5. Draw the graph of $y = x^2 - 4x + 5$. Can you say without drawing the graph, whether it cuts the X-axis or not? Why?
6. Draw the graph of $y = x^2 - 2x - 3$ and state the co-ordinates of the points of intersection of the graph with the X-axis.
7. What are the values of x that satisfy the equation $x^2 - 2x - 3 = 0$? Do you see any relation between the values of x that you have found and the co-ordinates of the points of intersection in Ex. 6 above? If there exists any relation, can you interpret it with respect to the graph?

§ 9.9 Solution of Quadratic Equations :

In Question 7 of Exercise 9.6 you were asked to note the relation between the values of the x -co-ordinates of the points of intersection of the graph of $y = x^2 - 2x - 3$ and the X-axis and the solutions of the equation $x^2 - 2x - 3 = 0$. You know that

the y -co-ordinate of every point on the X -axis is zero. The points on the graph of $y = x^2 - 2x - 3$, whose y -co-ordinates are zero, must be on the X -axis. This means, the x -co-ordinates of the points of intersection of $y = 0$ and $y = x^2 - 2x - 3$ are the solutions of the equation $x^2 - 2x - 3 = 0$. This is one of the methods by which we can find the solutions of quadratic equations graphically.

Moreover, when the solutions of quadratic equations are irrational numbers, we are interested in their approximate values, which can be obtained by graphs.

Let us consider the equation $x^2 + 2x - 5 = 0$.

This equation can be written as $(x^2 + 2x + 1) - 6 = 0$, i.e. $(x + 1)^2 - 6 = 0$.

If we draw the graph of $y = (x + 1)^2 - 6$ and read off the x -co-ordinates of the points of intersection of the graph with the X -axis, we shall get the solutions of the given equation.

For obtaining the algebraic solutions of the equation we proceed thus:

$$\begin{aligned} x^2 + 2x - 5 &= 0 \\ \therefore (x^2 + 2x + 1) - 6 &= 0 \\ \therefore (x + 1)^2 &= 6 \\ \therefore x + 1 &= \sqrt{6} \text{ or } x + 1 = -\sqrt{6} \\ \therefore x &= -1 + \sqrt{6} \text{ or } x = -1 - \sqrt{6} \end{aligned}$$

The solutions are irrational numbers and their rational approximations are 1.4 or -3.4 correct to one place of decimals.

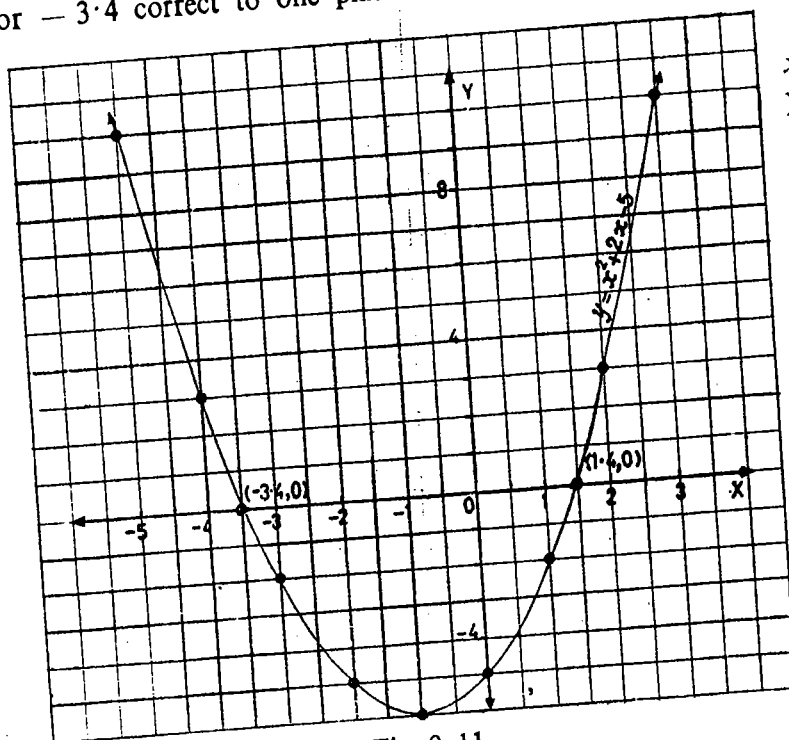


Fig. 9.11

The graph of $y = x^2 + 2x - 5$ cuts the X -axis in the points $(1.4, 0)$ and $(-3.4, 0)$. So, the graph gives us the approximate solutions (see fig. 9.11).

Another method of solving the above quadratic equation graphically would be to put it in the form $x^2 = 5 - 2x$ and then to equate each side of the equation to y , giving $y = x^2$ and $y = 5 - 2x$.

The intersections of the two graphs of $y = x^2$ and $y = 5 - 2x$ make the two expressions x^2 and $5 - 2x$ equal, which is the same as $x^2 + 2x - 5 = 0$.

So we draw the graphs of $y = x^2$ and $y = 5 - 2x$ and read off the x -co-ordinates of the points of intersection (see fig 9.12).

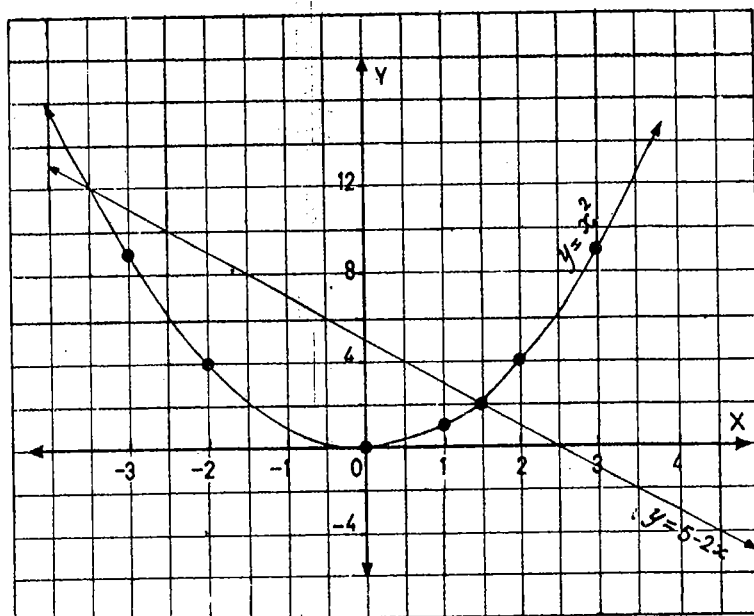


Fig. 9.12

EXERCISE 9.7

1. Solve the equation $x^2 - 4x - 6 = 0$ by the method of completing the square. Find its solutions correct to one place of decimals. Then draw the graph of $y = x^2 - 4x - 6$ putting the equation in the form $y = (x - 2)^2 - 10$ and read off the co-ordinates of the points of intersection where the graph cuts the X-axis. Verify whether the graphical solutions tally with the algebraic solutions.
2. Draw the graph of $y = x^2 - 4x + 1$. Write down the solutions of the equation $x^2 - 4x + 1 = 0$ from your graph. Verify your solutions solving the equation algebraically.
3. Solve the equation $x^2 + 2x - 4 = 0$ graphically.
4. Transform the equation $x^2 - 3x - 4 = 0$ as $x^2 = 3x + 4$ and put $y = x^2 = 3x + 4$. Draw the graphs of the two equations $y = x^2$ and $y = 3x + 4$ to solve the equation $x^2 - 3x - 4 = 0$.
5. Draw the graphs of $y = x^2$ and $y = 3x + 5$ and use them to solve the equation $x^2 - 3x - 5 = 0$.

§ 9.10 Quadratic Formula :

We have used the method of completing the square to solve quadratic equations. We shall use the said method to solve the general quadratic equation viz. $ax^2 + bx + c = 0$.

$ax^2 + bx + c = 0$. Since $a \neq 0$, dividing both sides by 'a', we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Completing the square, $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$

$$\therefore \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0 \quad \therefore \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\therefore x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \text{ or } x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\therefore x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The above are the solutions of the general quadratic equation in terms of the coefficients a , b and c .

The above solutions may be compactly written as $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

This is known as the **quadratic formula**. We can employ this formula for solving any quadratic equation. Consider the equation $x^2 - x - 2 = 0$. This equation compared with the general equation $ax^2 + bx + c = 0$, gives $a = 1$, $b = -1$, and $c = -2$. Substituting these values in the formula we get,

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2 \times 1} \\ &= \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2} = 2 \text{ or } -1. \end{aligned}$$

Example 1: Solve the equation $2x^2 - 3x - 5 = 0$ by the formula. Comparing, $2x^2 - 3x - 5 = 0$ with $ax^2 + bx + c = 0$.

We have, $a = 2$, $b = -3$ and $c = -5$.

$$\begin{aligned} \therefore x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{9 + 40}}{4} \\ &= \frac{3 \pm 7}{4} = \frac{5}{2} \text{ or } -1 \end{aligned}$$

$$\therefore x = \frac{5}{2} \text{ or } x = -1.$$

Example 2: Solve. $x^2 - 5x + 1 = 0$.

Here, $a = 1$, $b = -5$ and $c = 1$:

$$\begin{aligned} \therefore x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{25 - 4}}{2} \\ &= \frac{5 \pm \sqrt{21}}{2} \end{aligned}$$

$$\therefore x = \frac{5 + \sqrt{21}}{2} \text{ or } x = \frac{5 - \sqrt{21}}{2}$$

EXERCISE 9.8

Solve the following equations using the quadratic formula :

1. $x^2 - 2x - 3 = 0$
2. $x^2 - 4x - 5 = 0$
3. $2x^2 - x - 3 = 0$
4. $3x^2 = x + 5$
5. $4 - x^2 + 3x = 0$
6. Find a pair of numbers that differ by 3 and whose product is 108. Is there a second pair of numbers satisfying these conditions? If so, find that pair too.
7. The area of a rectangle is 120 square centimetres. Its length is 7 cm more than its breadth. Find its dimensions.
8. The sum of the altitude and the base of a triangle is 22 cm. Find the lengths of the base and the altitude over it, if the area of the triangle is 52.5 sq. cm.
9. In driving from Nagpur to Bombay a distance of 810 km a motorist discovered that he could make the trip in less time by 2 hours, increasing his speed by 13.5 km an hour. At what speed was he originally travelling ?
10. The expenses of a trip of a certain class amounted to Rs. 450. If five more students had accompanied the trip, the expenses per student would have been reduced by Re. 1. What was the number of students who joined the trip ?

§ 9.11 Nature of the Solution Set of Quadratic Equations :

We have seen that the solutions of the quadratic equation $ax^2 + bx + c = 0$, ($a \neq 0$) are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Since a, b, c are real numbers, so is $b^2 - 4ac$ a real number. This real number may be positive, negative or zero.

Observe the values of $b^2 - 4ac$ and the solutions in respect of the following equations.

Example 1 : $x^2 - 6x + 8 = 0$. Here we have, $a = 1, b = -6, c = 8$.

$$b^2 - 4ac = (-6)^2 - 4(1)(8) = 36 - 32 = 4$$

$$\begin{aligned} \text{Solutions} &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(8)}}{2(1)} \\ &= \frac{6 \pm \sqrt{4}}{2} = 4 \text{ and } 2. \end{aligned}$$

Example 2 : $x^2 - 6x + 9 = 0$. Here, $a = 1, b = -6, c = 9$.

$$b^2 - 4ac = (-6)^2 - 4(1)(9) = 36 - 36 = 0$$

$$\begin{aligned} \text{Solutions} &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(9)}}{2(1)} \\ &= \frac{6 \pm \sqrt{0}}{2} = 3 \text{ and } 3. \end{aligned}$$

Example 3 : $x^2 - 6x + 13 = 0$. Here, $a = 1$, $b = -6$, $c = 13$.

$$b^2 - 4ac = (-6)^2 - 4(1)(13) = 36 - 52 = -16$$

$$\begin{aligned}\text{Solutions} &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(13)}}{2(1)} \\ &= \frac{6 \pm \sqrt{-16}}{2}\end{aligned}$$

Now, answer the following questions with reference to the above examples.

1. What is the sign of $b^2 - 4ac$ in Ex. 1 ? Are the solutions real numbers ? If so, are they different or equal ?
2. Are the solutions in Ex. 2 equal or different ? Why are the solutions equal ?
3. Are the solutions in Ex. 3 real numbers ? What is the sign of $b^2 - 4ac$? State the reason why the solutions are not real numbers.
4. Do you notice any correspondence between the nature of the above solutions and the sign of $b^2 - 4ac$? If so, what is it ?

From the answers to the above questions you must have realized that the solutions of a quadratic equation may be : (1) two distinct real numbers ; or (2) two equal real numbers ; or (3) not real numbers ; and that this nature of the solutions depends upon the expression $b^2 - 4ac$.

1. If $b^2 - 4ac$ is a positive real number, then its square root is a real number, and hence, the solutions of the quadratic equation are real numbers which are distinct.
2. If $b^2 - 4ac$ is zero, its square root is also zero. In this case, the solutions of the quadratic equation are equal. We also say that the quadratic equation has only one solution. In such a case, the expression $ax^2 + bx + c$ is a perfect square.
3. If $b^2 - 4ac$ is a negative real number, its square root is not a real number and hence the solutions of the quadratic equation will not be real numbers. In other words, the quadratic equations for which $b^2 - 4ac$ is negative, have their solution-sets empty.

We can discriminate about the nature of the solutions of a quadratic equation according as $b^2 - 4ac$ is positive, zero or negative. Hence, the expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$.

It should be borne in mind that $\sqrt{b^2 - 4ac}$ is not called the discriminant. When we are only concerned with the nature of the roots of a quadratic equation, we need not bother ourselves to know the actual solutions and we may use the discriminant for our purpose.

Example 4 : $2x^2 - 7x + 5 = 0$.

$$\begin{aligned}\text{Discriminant} &= b^2 - 4ac = (-7)^2 - 4(2)(5) \\ &= 49 - 40 = 9\end{aligned}$$

$$\text{Solutions} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(5)}}{2(2)}$$

$$= \frac{7 \pm \sqrt{9}}{4} = \frac{5}{2} \text{ and } 1.$$

Example 5 : $x^2 - 8x + 9 = 0$.

$$\begin{aligned}\text{Discriminant} &= b^2 - 4ac = (-8)^2 - 4(1)(9) \\ &= 64 - 36 = 28\end{aligned}$$

$$\text{Solutions} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(9)}}{2(1)}$$

$$= \frac{8 \pm \sqrt{28}}{2} = 4 + \sqrt{7} \text{ and } 4 - \sqrt{7}.$$

In the above examples a, b, c are rational numbers. In Example 4, $b^2 - 4ac = 9$ which is the square of the rational number 3, and the solutions are also rational numbers. In contrast, we have in Ex. 5, $b^2 - 4ac = 28$, which is not the square of a rational number. $\sqrt{b^2 - 4ac} = \sqrt{28}$ and it is an irrational number. The solutions are also irrational numbers and they are binomial conjugate surds, which fact you must have noticed !

From the above examples, we can say, that if a, b, c are rational numbers and if $b^2 - 4ac$ is a positive number, then there are two possibilities. $b^2 - 4ac$ may be a perfect square of a rational number or it may be a square of an irrational number. If $b^2 - 4ac$ is the square of a rational number then the solutions are rational numbers and if $b^2 - 4ac$ is a square of an irrational number, then the solutions are irrational numbers and they are binomial conjugate surds.

If we know that the coefficients in a given quadratic equations are rational numbers and that one solution is $3 + \sqrt{5}$, then we can say that the other solution is $3 - \sqrt{5}$, which the binomial conjugate surd of the given solution.

Example : Discuss the nature of the solutions of the following quadratic equations with the help of the discriminant (i.e. state whether the solutions are distinct real numbers or equal real numbers or are not real numbers).

(i) $3x^2 - 5x - 2 = 0$ (ii) $4x^2 + 20x + 25 = 0$ (iii) $2x^2 - 3x + 5 = 0$

(i) In this equation, $a = 3$, $b = -5$, $c = -2$,

$$\therefore \text{Discriminant} = b^2 - 4ac = (-5)^2 - 4(3)(-2), \\ = 25 + 24 = 49. \text{ This is a positive real number.}$$

$\therefore$ The solutions of this equation are distinct real numbers.

(ii) In this equation, $a = 4$, $b = 20$, $c = 25$.

$$\therefore \text{Discriminant} = b^2 - 4ac = (20)^2 - 4(4)(25). \\ = 400 - 400 = 0.$$

$\therefore$ The solutions of this equation are equal real numbers.

(iii) In this equation $a = 2$, $b = -3$, $c = 5$.

$$\therefore \text{Discriminant} = b^2 - 4ac = (-3)^2 - 4(2)(5) \\ = 9 - 40 = -31. \text{ This is a negative real number.}$$

$\therefore$ The solutions of this equation are not real numbers.

EXERCISE 9.9

Decide the nature of the solutions of the following equations with the help of the discriminant.

1. $x^2 - 8x + 16 = 0$
2. $8x + 5 = x^2$
3. $x^2 = 12x - 36$
4. $2x^2 - 3x + 4 = 0$
5. $3x^2 - 8x + 5 = 0$
6. $5x^2 - 6x + 3 = 0$
7. $9 - 6x - x^2 = 0$
8. $4x^2 + 13x + 3 = 0$
9. $\frac{1}{2}x^2 + 3x + \frac{9}{2} = 0$
10. $\frac{3}{4}x^2 + 2x + \frac{3}{4} = 0$
11. $x^2 + x + 1 = 0$

[In Examples 9 and 10, first multiply both the sides of the equations by the lowest common multiple of the denominators of the terms involved].

Find the solutions of the following equations with the help of the quadratic formula and decide whether they are rational or irrational. How will you decide whether the solutions are rational or irrational?

12. $x^2 = x + 6$
13. $x^2 + 4 = 6x$
14. $x^2 + \frac{x}{2} = 3$
15. $x^2 + 4x + 1 = 0$
16. $6x^2 = x + 1$
17. $x^2 + 4x + \frac{11}{4} = 0$

Chapter 10 : RATIO AND PROPORTION

§ 10.1 Comparison of Two Quantities :

We are often required to compare two quantities of the same kind. This comparison can be done in two ways. Consider the following example. In an examination in Mathematics A and B secured 66 and 44 marks respectively. We compare their marks as follows.

(1) We may say, A obtained 22 marks more than B;

or (2) We can say, A's marks are $1\frac{1}{2}$ times that of B. We are now interested in the second type of comparison.

When we say that one quantity is 'so many times' the other, we are stating the ratio of the first quantity to the second.

In the above example, the ratio of A's marks to B's marks is 66 : 44 or 3 : 2.

To express the ratio of one quantity to any other, we divide the first quantity by the second one. The ratio of 6 to 9 is written in one of the following ways.

$$6 \div 9 \text{ or } \frac{6}{9} \text{ or } 6 : 9.$$

If x and y are positive real numbers, then their ratio is written as $\frac{x}{y}$, or $x \div y$, or $x : y$.

Suppose P and M possess amounts of Rs. 15 and Rs. 10 respectively. Then the ratio of the amount possessed by P to that by M is 15 : 10 and not Rs. 15 : Rs. 10. Note that the quantities to be compared must be in the same units of measurement.

In general, if two quantities measure x and y units of the same kind, the ratio of the two quantities is $\frac{x}{y}$. Thus a ratio of two quantities is the comparison of their magnitudes when expressed in the same unit.

§ 10.2 Terms of a Ratio :

The numbers which form a ratio are called the **terms** of the ratio. The ratio of the number ' r ' to the number ' s ' is the fraction $\frac{r}{s}$. In this ratio ' r ' is called the **antecedent** and ' s ' is called the **consequent**.

You may recall from your knowledge of working with fractions, that if x, y are any positive real numbers, then the fraction $\frac{x}{y}$ is the same as $\frac{kx}{ky}$ and hence the ratio $\frac{x}{y}$ is the same as $\frac{kx}{ky}$. Using this fact the ratio $\frac{x}{y}$ is generally expressed as $\frac{x}{y}$ and then it is said to be in its reduced fractional form.

§ 10.3 Magnitude and Ratio :

We have seen that a ratio is a comparison of two quantities of the same kind. It would be just ridiculous to compare Ashok's weight with Alka's age. We can compare only those quantities which are of the same kind. Can we compare Rs. 2 with 27 paise? Yes! this is permissible, provided we put the two quantities in the same units. Rs. 2 is 200 paise and so the ratio of Rs. 2 to 27 paise is 200 : 27 and not 2 : 27. Of the two given quantities of the same kind, but expressed in different units, we must convert one of them in the same units as that of the other. For instance, suppose we wish to find the ratio of the lengths of two poles whose lengths are 4 metres and 80 cm respectively. We must change the units of measurement of one of them into the units of measurement of the other. Thus we convert 80 cm into 0.8 metres and then the ratio of the two lengths in question would be 4 : 0.8 which is the same as 40 : 8 or 5 : 1. Alternatively we may convert 4 metres into 400 cm, and then the ratio would be 400 : 80 which reduces to 5 : 1. Will the ratio of these lengths, if expressed in decimetres, remain the same? Just verify.

A's purse contains an amount of Rs. 4 and my purse contains 53 paise. The ratio of the two amounts is not 4 : 53. Why? What is the correct ratio?

EXERCISE 10.1

1. In the following examples express the ratio of the first quantity to the second. State the conditions on the variables if necessary.

(i) 3, 9 (ii) 2, 7 (iii) 9 cm, 15 cm (iv) $4x^2$, $7x^2$ (v) 55 cm, 2 metres
(vi) 77 paise, Rs. 11 (vii) x , $2y$ (viii) 6 yrs. 2 months, 64 yrs. 9 months
(ix) 45 seconds, one hour (x) $\sqrt{2}$, 2 (xi) $2\sqrt{24}$, $3\sqrt{2}$ (xii) $x\sqrt{2}$, y^2 .

2. If two numbers are in the ratio 2 : 3, we can express it as $\frac{2x}{3x}$, where x is a positive real number.

The ages of Ashok and his father are in the ratio 3 : 8. If the difference in their ages is 35 years, find Ashok's age. [You already know how to solve word problems. The only thing that is new to you is the idea of ratios. Use this idea to solve this problem. You can proceed thus. The ages of Ashok and his father are in the ratio 3 : 8. Hence, let their ages be $3x$ years and $8x$ years, respectively. Form an equation from the given condition and solve the problem.]

3. Two numbers are in the ratio 3 : 5 and their sum is 360. Find them.
4. Two numbers are in the ratio $a : b$. The second one is x . What is the first?
5. The ages of Kavita and Salma are in the ratio 3 : 4. In four years' time their ages will be in the ratio 4 : 5. Find their present ages.
6. When 1 is added to each of two given numbers, their ratio becomes 1 : 2. When 5 is subtracted from each of them the ratio becomes 5 : 11. Find the numbers.
7. The volumes of two cubes are in the ratio 125 : 27. Find the ratio of the areas of their surfaces.
8. In an isosceles triangle the length of one of the congruent sides is two-thirds that of its base. Find the ratio of the length of the base to that of one of the congruent sides. Then find the length of the base of the isosceles triangle given that its perimeter is 84 cm.

9. The sides of a rectangle are in the ratio 5 : 3 and the area of the rectangle is 29.4 sq. cm. Find the length and the breadth of the rectangle.
10. The measures of the angles A and B of the parallelogram ABCD are in the ratio 2 : 7. Find the measure of angle D.
11. The measures of the angles of the quadrilateral ABCD are in the ratio 2 : 3 : 4 : 1, which of the sides of the quadrilateral are parallel?
12. The measures of the radial segments of two circles are in the ratio 2 : 3. The measure of the radial segment of the smaller circle is 3.5 cm. Find the area of the greater circle.
13. If the radius of a circle be increased by 10%, what will be the corresponding increase (percent) in its area?

§ 10.4 Classification of Ratios :

Consider the following ratios : $\frac{5}{7}$, $\frac{13}{23}$, $\frac{1\frac{2}{3}}{5\frac{3}{4}}$.

All these ratios can be expressed in the form $a : b$ where a and b are positive integers. They can be represented either by terminating or recurring decimals.

We call such ratios **commensurable**. Ratios like $\frac{\sqrt{3}}{\sqrt{5}}$, $\frac{1}{\sqrt{2}}$, which cannot be expressed as recurring or terminating decimals, are called **incommensurable** ratios.

§ 10.5 Comparison of Ratios :

From the above discussion it is quite obvious that there is no difference substantially between a positive fraction and a ratio. Hence, the operations that can be carried out on fractions can be carried out in the same manner on ratios.

We know that if the numerator and the denominator of a fraction be both multiplied by or divided by the same non-zero number, we get an equivalent fraction; that is, the value of the fraction remains unchanged. This property equally holds for ratios, and the same can be used to compare two or more ratios, i. e. to determine the equality or inequality relation between them. Study the following example carefully and justify each step.

REASON

$$\begin{array}{lcl} \frac{3}{5} = \frac{3}{5} \times 1 & \text{—————} & \frac{7}{12} = \frac{7}{12} \times 1 \\ & & \\ & = \frac{3}{5} \times \frac{12}{12} & \text{—————} & = \frac{7}{12} \times \frac{5}{5} \\ & = \frac{36}{60} & \text{—————} & = \frac{35}{60} \end{array}$$

We know that $\frac{36}{60} > \frac{35}{60}$. (How ?)

$$\therefore \frac{3}{5} > \frac{7}{12} \text{ or } \frac{7}{12} < \frac{3}{5}$$

In the above example if we take $a = 3$, $b = 5$, $c = 7$, $d = 12$, then $\frac{a}{b} > \frac{c}{d}$,
and $ad = 3 \times 12 = 36$ and $bc = 5 \times 7 = 35$. $\therefore ad > bc$.

Hence we can conclude that if $\frac{a}{b} > \frac{c}{d}$, then $ad > bc$, where a, b, c, d are positive real numbers.

Can we conclude that if a, b, c, d are positive real numbers and if $ad > bc$, then $\frac{a}{b}$ is always greater than $\frac{c}{d}$?

If your answer to the above question is in the affirmative, then you are right. Let us see if we can prove it.

$$ad > bc \quad \dots (\text{given})$$

a, b, c, d are positive real numbers. $\therefore bd$ is a positive real number.

Dividing both sides of the above inequality by the positive real number bd , we get

$$\frac{ad}{bd} > \frac{bc}{bd}$$

$$\therefore \frac{a}{b} > \frac{c}{d} \quad \dots (\text{reducing the fractions}).$$

EXERCISE 10.2

1. Which of the following ratios are commensurable and which are incommensurable?

(i) $\sqrt{8} : \sqrt{162}$

(ii) $4\sqrt{3} : \sqrt[3]{27}$

(iii) $\sqrt{72} : \sqrt{18}$

(iv) $\sqrt{12} : \sqrt{18}$

(v) $\frac{\sqrt{5}x^2}{\sqrt{3}x^2} \quad (x \neq 0)$

(vi) $\frac{3\sqrt{2}x^2}{5\sqrt{2}x} \quad (x > 0)$

2. Fill in the blanks with ' $>$ ' or ' $<$ ' or ' $=$ ' to make the following statements true.

(i) $\frac{3}{4} \dots \frac{7}{9}$

(ii) $\frac{4}{7} \dots \frac{5}{6}$

(iii) $\frac{2.5}{7} \dots \frac{3}{8}$

(iv) $\frac{4}{11} \dots \frac{36}{99}$

(v) $\frac{3\sqrt{3}}{2\sqrt{2}} \dots \frac{2\sqrt{2}}{3\sqrt{3}}$

(vi) $\left(\frac{1}{3}\right)^5 \dots \left(\frac{1}{5}\right)^3$

3. ABCD is a square whose side is x units. What is the length of BD? Find the ratio of the lengths of BD and AB. Is the ratio commensurable?
4. Sometimes it is advantageous to put a given ratio in the form $k : 1$, i.e. a form in which the second term is unity (one). For instance, $3 : 4$ is written as $\frac{3}{4} : 1$. Write the following ratios in the form $k : 1$.

(i) $5 : 7$

(ii) $\frac{7}{5} : \frac{3}{4}$

(iii) $6 : \sqrt{3}$

(iv) $4 : \sqrt{8}$

(v) $6 : 2\sqrt{3}$

5. If $x : y = 2 : 3$ and $y : z = 5 : 7$, find the ratio $x : z$.

§ 10.6 Proportion :

$a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3, \dots$ are two sequences of positive real numbers.

If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots$, then we say that the two sequences are **proportional**.

If $\frac{a_1}{b_1} = k$, then we conclude that

$$\frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots = k.$$

From the equations $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots = k$, we observe that $a_1 = b_1 k$, $a_2 = b_2 k$, $a_3 = b_3 k$, and so on.

The proportionality with which we are interested at present is that which involves only four numbers. This proportionality is generally known as a **proportion**.

If a, b, c , and d are four positive real numbers such that $\frac{a}{b} = \frac{c}{d}$, then we say that a, b, c, d are **in proportion**. The numbers a, b, c, d are called the **terms** of the proportion; ' d ' is said to be the **fourth proportional** to a, b and c . The first and the fourth numbers of a proportion are known as the **extremes** while the second and the third are called the **means**. Thus, we see that a proportion is a statement of equality of two ratios — the first ratio being the ratio of the first term to the second and the second ratio being the ratio of the third term to the fourth.

The statements $\frac{a+1}{b} = \frac{c}{d}$, $\frac{x+3}{5-y} = \frac{1+z}{xy}$ are proportions but the statement $\frac{a}{b} + 1 = \frac{c}{d}$ is not a proportion as it is not in the form of a proportion.

The term 'proportion' is, therefore, with reference to a particular form. Note that the statement $\frac{a}{b} + 1 = \frac{c}{d}$ is equivalent to $\frac{a+b}{b} = \frac{c}{d}$, and in that form it is a 'proportion'.

EXERCISE 10.3

1. State whether each of the following statements is true or false.

(i) $1 : 2 = 6 : 12$

(ii) $5 : 7 = 15 : 21$

(iii) $\frac{4}{7} = \frac{15}{21}$

(iv) $\frac{2}{7} = \frac{6}{21} = \frac{8}{28}$

(v) $\frac{6}{10} = \frac{6}{12} = \frac{10}{12}$

(vi) $\frac{4}{16} = \frac{2}{8} = \frac{3}{12}$

2. Fill in the blanks in the following statements making them true.

$$(i) \frac{2}{\dots\dots} = \frac{3}{15} = \frac{4}{\dots\dots} \quad (ii) \frac{10}{60} = \frac{5}{\dots\dots} = \frac{3}{\dots\dots}$$

$$(iii) \frac{7}{14} = \frac{17}{\dots\dots} = \frac{27}{\dots\dots} = \frac{9}{\dots\dots} = \frac{\dots\dots}{4}$$

$$(iv) \frac{1}{3} = \frac{2}{\dots\dots} = \frac{\dots\dots}{15} = \frac{\dots\dots}{105} = \frac{37}{\dots\dots}$$

3. Find x so that each statement given below is a proportion.

$$(i) \frac{7}{x} = \frac{35}{15} \quad (ii) \frac{15}{20} = \frac{x}{4} \quad (iii) \frac{12}{8} = \frac{x}{6}$$

$$(iv) \frac{5}{3} = \frac{x}{21} \quad (v) \frac{5}{x} = \frac{3}{4}$$

4. Complete the following statements making them true.

$$(i) \text{ If } 7x = 3y, \text{ then } \frac{x}{y} = \dots\dots \text{ and } \frac{x}{3} = \dots\dots$$

$$(ii) \text{ If } 3a = 5b, \text{ then } \frac{3}{b} = \dots\dots \text{ and } \frac{b}{a} = \dots\dots$$

$$(iii) \text{ If } 6x = 9 \times 2, \text{ then } \frac{x}{9} = \dots\dots \text{ and } \frac{2}{x} = \dots\dots$$

$$(iv) \text{ If } ad = bc, \text{ then } \frac{a}{b} = \dots\dots \text{ and } \frac{a}{c} = \dots\dots$$

$$(v) \text{ If } \frac{3a}{4b} = \frac{6c}{12d}, \text{ then } \frac{a}{b} = \dots\dots \text{ and } \frac{d}{c} = \dots\dots$$

§ 10.7 Some Results about Proportion :

We have seen that the terms of a ratio are positive real numbers and we use the properties of real numbers while solving problems involving ratios.

If x, y, z and w are positive real numbers and $x = y$ and $z = w$, then we know that $xz = yw$. Making use of this property of real numbers we can establish a simple but important result about a proportion.

Theorem of the Means and the Extremes—

If a, b, c and d are positive real numbers such that $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$.

$$\text{Proof : (1) } \frac{a}{b} = \frac{c}{d}$$

$$(2) \frac{a}{b} \times bd = \frac{c}{d} \times bd$$

$$(3) ad = bc$$

We can state the above result in words as follows.

In a proportion, the product of the extremes is equal to the product of the means.

Now we deduce a number of results about proportions which are found useful in computations.

(I) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{b}{a} = \frac{d}{c}$.

Proof : $\frac{a}{b} = \frac{c}{d} \therefore ad = bc$

$$\therefore \frac{ad}{ca} = \frac{bc}{ca}$$

$$\therefore \frac{d}{c} = \frac{b}{a}$$

$$\therefore \frac{b}{a} = \frac{d}{c}$$

This result is called **invertendo**.

(II) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{c} = \frac{b}{d}$.

Proof : $\frac{a}{b} = \frac{c}{d} \therefore ad = bc$

$$\therefore \frac{ad}{cd} = \frac{bc}{cd}$$

$$\therefore \frac{a}{c} = \frac{b}{d}$$

This result is known as **alternando**.

(III) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{b} = \frac{c+d}{d}$.

Proof : $\frac{a}{b} = \frac{c}{d}$

$$\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1$$

$$\therefore \frac{a+b}{b} = \frac{c+d}{d}$$

This result is called **componendo**.

(IV) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a-b}{b} = \frac{c-d}{d}$.

Proof : [This may be supplied by the pupils on the lines of (III) above.]

This result is known as **dividendo**.

(V) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$, provided $\frac{a}{b} = \frac{c}{d} \neq 1$.

Proof : $\frac{a}{b} = \frac{c}{d}$

$$\therefore \frac{a+b}{b} = \frac{c+d}{d} \quad \dots (\text{componendo})$$

$$\text{Also } \frac{a-b}{b} = \frac{c-d}{d} \quad \dots (\text{dividendo})$$

$$\therefore \frac{b}{a-b} = \frac{d}{c-d}$$

$$\therefore \frac{a+b}{b} \times \frac{b}{a-b} = \frac{c+d}{d} \times \frac{d}{c-d}$$

$$\therefore \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

This result is known as **componendo-dividendo**.

The above results are useful in reducing the working in some problems.

Example 1: If $\frac{x}{y} = \frac{5}{3}$, find the value of $\frac{3x+2y}{3x-2y}$.

$$\frac{x}{y} = \frac{5}{3} \quad \therefore \frac{x}{5} = \frac{y}{3} = k \quad \dots (\text{alternando})$$

$$\therefore x = 5k \text{ and } y = 3k.$$

$$\begin{aligned} \therefore \frac{3x+2y}{3x-2y} &= \frac{3 \times 5k + 2 \times 3k}{3 \times 5k - 2 \times 3k} = \frac{15k + 6k}{15k - 6k} \\ &= \frac{21k}{9k} = \frac{7}{3}. \end{aligned}$$

Alternative Method :

$$\frac{x}{y} = \frac{5}{3}$$

$$\therefore \frac{3}{2} \times \frac{x}{y} = \frac{3}{2} \times \frac{5}{3} \quad \therefore \frac{3x}{2y} = \frac{5}{2}$$

$$\therefore \frac{3x+2y}{3x-2y} = \frac{5+2}{5-2} \quad \dots (\text{componendo-dividendo})$$

$$\therefore \frac{3x+2y}{3x-2y} = \frac{7}{3}$$

[Note : In such cases it is usually advisable to use 'componendo-dividendo' result. That explains why we multiplied both sides of $\frac{x}{y} = \frac{5}{3}$ by $\frac{3}{2}$.]

Example 2 : Solve the equation. $\frac{(3x+2)^3 + (3x-2)^3}{(3x+2)^3 - (3x-2)^3} = \frac{9}{7}$.

Part 1 : $\frac{(3x+2)^3 + (3x-2)^3}{(3x+2)^3 - (3x-2)^3} = \frac{9}{7}$ (given)

$$\therefore \frac{[(3x+2)^3 + (3x-2)^3] + [(3x+2)^3 - (3x-2)^3]}{[(3x+2)^3 + (3x-2)^3] - [(3x+2)^3 - (3x-2)^3]} = \frac{9+7}{9-7}$$

$$\therefore \frac{2(3x+2)^3}{2(3x-2)^3} = \frac{16}{2} \quad \therefore \frac{(3x+2)^3}{(3x-2)^3} = \frac{8}{1}$$

Taking cube-roots of both sides, $\frac{3x+2}{3x-2} = \frac{2}{1}$

$$\therefore \frac{(3x+2) + (3x-2)}{(3x+2) - (3x-2)} = \frac{2+1}{2-1} \quad \dots \text{ (componendo-dividendo)}$$

$$\therefore \frac{6x}{4} = \frac{3}{1} \therefore x = 2.$$

Part 2 : Substituting 2 for x in the given equation,

$$\frac{(3 \times 2 + 2)^3 + (3 \times 2 - 2)^3}{(3 \times 2 + 2)^3 - (3 \times 2 - 2)^3} \stackrel{?}{=} \frac{9}{7}$$

$$\text{i.e. } \frac{8^3 + 4^3}{8^3 - 4^3} \stackrel{?}{=} \frac{9}{7} \quad \text{i.e. } \frac{512 + 64}{512 - 64} \stackrel{?}{=} \frac{9}{7}$$

$$\text{i.e. } \frac{576}{448} \stackrel{?}{=} \frac{9}{7} \quad \text{i.e. } \frac{9}{7} \stackrel{\checkmark}{=} \frac{9}{7}$$

$\therefore$ The solution set of the given equation is $\{2\}$.

Example 3 : Solve. $\frac{x^2 - 3x + 2}{3x - 2} = \frac{x^2 - 5x + 4}{5x - 4}$

Part 1 : Observe that the left-hand side of the equation contains $-(3x - 2)$ in the numerator and $(3x - 2)$ in the denominator, while in the right-hand side $-(5x - 4)$ and $(5x - 4)$ appear in the numerator and in the denominator respectively. Hence, we can simplify our computation by using componendo.

$$(1) \quad \frac{x^2 - 3x + 2}{3x - 2} = \frac{x^2 - 5x + 4}{5x - 4} \quad (\text{given})$$

$$\therefore (2) \quad \frac{x^2}{3x - 2} = \frac{x^2}{5x - 4} \quad (\text{By componendo})$$

$$\therefore (3) \quad \text{Either } x^2 = 0 \text{ or } 3x - 2 = 5x - 4.$$

$$\therefore (4) \quad x = 0 \text{ or } x = 1.$$

[Note : In step (3) above, we have concluded that either $x^2 = 0$ or $3x - 2 = 5x - 4$. Note this step carefully. In step (2) we have two equal expressions whose numerators are equal but denominators are different. So, two cases arise: (i) If the numerators are zero, the equality is true ; or (ii) if the numerators are not zero, then for the equality to be true, the denominators must also be equal.]

Part 2 : Verification is left as an exercise for the students.

EXERCISE 10.4

1. Find the fourth proportional to—

(i) 12, 9, 32. (ii) 2, 6, 7. (iii) 4, 12, 5. (iv) ab , bc , cd . (v) $6a^2b$, $9a^3$, $8b^3$.

2. If $\frac{a}{b} = \frac{3}{4}$ and $\frac{b}{c} = \frac{4}{5}$, find the value of $\frac{a}{c}$.

3. Complete the following statements.

(i) $\frac{3}{4} = \frac{15}{20} \therefore \frac{4}{3} = \dots$ and $\frac{3}{15} = \dots$

(ii) $\frac{11}{5} = \frac{99}{45} \therefore 11 \times 45 = \dots \times \dots$ and $\frac{11}{99} = \dots$

(iii) $\frac{7}{3} = \frac{63}{27} \therefore \frac{7+3}{3} = \dots$ and $\frac{\dots}{3} = \frac{63-27}{27}$

(iv) $\frac{13}{17} = \frac{39}{51} \therefore \frac{13}{17-13} = \frac{39}{\dots}$

(v) If $\frac{a}{b} = \frac{17}{11}$ then $\frac{a+b}{b} = \dots$ and $\frac{a-b}{b} = \dots$

(vi) If $\frac{x}{y} = \frac{5}{3}$ then $\frac{x+y}{x-y} = \dots$

(vii) If $\frac{x}{y} = \frac{11}{5}$ then $\frac{5x}{2y} = \dots$ and $\frac{5x+2y}{5x-2y} = \dots$

(viii) If $\frac{2x}{3y} = \frac{6}{15}$ then $\frac{x}{y} = \dots$ and $\frac{x+y}{y-x} = \dots$

(ix) If $8x - 15y = 0$ then $\frac{x-y}{x+y} = \dots$

(x) If $\frac{3x+y+5}{7x-y+3} = \frac{5}{3}$ then $x:y = \dots : \dots$

4. Complete the following statements.

(i) If $\frac{a^2+b^2}{a^2-b^2} = \frac{25}{7}$ then $\frac{a+b}{a-b} = \dots$

(ii) If $\frac{3x+2y}{3x-2y} = \frac{25}{17}$ then $\frac{x-y}{x+y} = \dots$

(iii) If $\frac{3x+5y}{3x-5y} = 11$ then $\frac{x}{y} = \dots$

(iv) If $\frac{2x^2+3y^2}{2x^2-3y^2} = \frac{125}{71}$ then $\frac{x}{y} = \dots$

5. Solve the following equations.

$$(i) \frac{(3x+1)^2 + (x-1)^2}{(3x+1)^2 - (x-1)^2} = \frac{25}{24} \quad (ii) \frac{x^2 - 7x + 5}{7x - 5} = \frac{x^2 - 5x - 3}{5x + 3}$$

6. If $x = \frac{2ab}{a+b}$ find the value of $\frac{x+a}{x-a} + \frac{x+b}{x-b}$.

$$\left(\text{Hint : } x = \frac{2ab}{a+b} \therefore \frac{x}{a} = \frac{2b}{a+b} \right).$$

§ 10.8 Continued Proportion :

If a, b, c are three positive real numbers such that a, b and b, c are in proportion, that is to say, $\frac{a}{b} = \frac{b}{c}$, then a, b, c are said to be in **continued proportion**.

In this case, ' c ' is said to be the **third proportional** to a and b , and ' b ' is called the **mean proportional** between a and c , while ' a ' and ' c ' are called the **extremes**.

If a, b, c are in continued proportion, then we have $\frac{a}{b} = \frac{b}{c}$. That is, $b^2 = ac$.

Hence, if three numbers are in continued proportion, the square of the mean is equal to the product of the extremes.

The notion of continued proportion is not restricted to three numbers only. It can be extended to any number or numbers. If $a, b, c, d, e, \dots$ are such that $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \dots$, then $a, b, c, d, e, \dots$ are said to be in '**continued proportion**'.

If we equate each of the ratios in a continued proportion to ' k ', then we can do with only two variables and express all the numbers in continued proportion in terms of those two variables only. If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = k$, then, $d = ek, c = dk = ek^2, b = ck = ek^3$ and $a = bk = ek^4$. Thus, instead of a, b, c, d, e we may have ek^4, ek^3, ek^2, ek and e .

Example 4: Find the mean proportional between $9x^2$ and $\frac{1}{81x^4}$.

Let ' a ' be the mean proportional between $9x^2$ and $\frac{1}{81x^4}$.

$$\therefore \frac{9x^2}{a} = \frac{a}{\frac{1}{81x^4}} \therefore a^2 = 9x^2 \times \frac{1}{81x^4} = \frac{1}{9x^2}$$

$$\therefore a = \frac{1}{3x}$$

$$\therefore \frac{1}{3x} \text{ is the mean proportional between } 9x^2 \text{ and } \frac{1}{81x^4}.$$

Example 5: Find the third proportional to $x^2 - y^2$ and $x + y$. Let the third proportional be 'a'.

$$\therefore \frac{x^2 - y^2}{x + y} = \frac{x + y}{a}$$

$$\therefore (x + y)^2 = a(x^2 - y^2)$$

$$\therefore a = \frac{(x + y)^2}{x^2 - y^2} = \frac{(x + y)(x + y)}{(x + y)(x - y)} = \frac{x + y}{x - y}$$

$\therefore$ The required third proportional is $\frac{x + y}{x - y}$.

Example 6: If a, b, c are in continued proportion, prove that

$$a^2 b^2 c^2 \left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \right) = a^3 + b^3 + c^3.$$

$$\frac{a}{b} = \frac{b}{c} \quad \dots \text{(given)}$$

$$\therefore b^2 = ac$$

$$\begin{aligned} \text{Now, L. H. S.} &= a^2 b^2 c^2 \left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \right) \\ &= a^2 \times ac \times c^2 \left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \right) \\ &= a^3 c^3 \left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \right) \\ &= c^3 + \frac{a^3 c^3}{b^3} + a^3 = c^3 + \frac{(b^2)^3}{b^3} + a^3 \\ &= c^3 + b^3 + a^3 \\ \therefore \text{R. H. S.} &= a^3 + b^3 + c^3. \end{aligned}$$

Example 7: If a, b, c are in continued proportion, prove that

$$(a + b + c)(b - c) = ab - c^2.$$

If we expand the left-hand side of the statement to be proved, we get the expression $ab - ac + b^2 - c^2$. On the right-hand side, the expression contains only $ab - c^2$. The two terms $-ac + b^2$ on the left-hand side are together equal to zero, as $b^2 = ac$. So in the left-hand side we put ac for b^2 and get the required result.

Example 8: If a, b, c are in continued proportion, prove that

$$\frac{abc(a + b + c)^3}{(ab + bc + ca)^3} = 1.$$

As a, b, c are in continued proportion, we have $b^2 = ac$.

If we substitute b^2 for ac in the left-hand side of the statement to be proved, we shall get,

$$\begin{aligned} \text{L. H. S.} &= \frac{abc(a+b+c)^3}{(ab+bc+ca)^3} = \frac{b \times b^2(a+b+c)^3}{(ab+bc+b^2)^3} \\ &= \frac{b^3(a+b+c)^3}{[b(a+c+b)]^3} \\ &= \frac{b^3(a+b+c)^3}{b^3(a+b+c)^3} \end{aligned}$$

$$\therefore \text{L. H. S.} = \text{R. H. S.} = 1$$

Example 9: Find five numbers in continued proportion such that the product of the first and the fifth is 81 and the sum of the second and the fourth is 30.

Let a, b, c, d and e be the required numbers. As they are in continued proportion, we have, $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e}$.

Let us equate each of the above ratios to ' k '.

$$\text{Then } \frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = k, \text{ giving}$$

$$d = ek, c = dk = ek^2,$$

$$b = ck = ek^3, \text{ and } a = bk = ek^4.$$

Thus, instead of taking the five variables, a, b, c, d and e for the required numbers, we can do with only two variables e and k , representing the five numbers in continued proportion as ek^4, ek^3, ek^2, ek and e .

Now, according to the first condition in the problem we have,

$$(ek^4) \times (e) = 81 \quad \dots\dots\dots (1)$$

and according to the second condition, we have

$$ek^3 + ek = 30 \quad \dots\dots\dots (2)$$

From (1), we have $e^2k^4 = 81$

$$\therefore ek^2 = 9 \quad \therefore e = \frac{9}{k^2} \quad \dots\dots\dots (3)$$

Substituting this value of ' e ' in (2), we get

$$\frac{9}{k^2} \times k^3 + \frac{9}{k^2} \times k = 30$$

$$\therefore 9k + \frac{9}{k} = 30 \quad \therefore 9k^2 - 30k + 9 = 0 \quad \therefore 3k^2 - 10k + 3 = 0$$

$$\therefore (3k - 1)(k - 3) = 0 \quad \therefore 3k - 1 = 0 \text{ or } k - 3 = 0$$

$$\therefore k = \frac{1}{3} \text{ or } k = 3$$

$$\therefore \text{From (3), } e = \frac{9}{(\frac{1}{3})^2} \text{ or } e = \frac{9}{(3)^2}$$

$$\therefore e = 81 \text{ or } e = 1.$$

Choosing the corresponding values of k and e , namely $\frac{1}{3}$ and 81 respectively, the required numbers are

$$ek^4 = \frac{81}{81} = 1, \quad ek^3 = \frac{81}{27} = 3, \quad ek^2 = \frac{81}{9} = 9,$$

$$ek = \frac{81}{3} = 27 \quad \text{and} \quad e = 81.$$

$\therefore$ The required numbers are 1, 3, 9, 27 and 81.

If we take the other set of the corresponding values of e and k , namely $e = 1$ and $k = 3$, we shall get the same numbers, but in the reversed order. Verify this.

EXERCISE 10.5

1. Fill in the blanks.

(i) The mean proportional between $a^2 - \frac{1}{b^2}$ and $b^2 - \frac{1}{a^2}$ is

(ii) The mean proportional between $\frac{36a^3}{b^3}$ and $\frac{b}{9a}$ is

(iii) The mean proportional between $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ is

(iv) The third proportional to x^2 and xy is

(v) The third proportional to $\frac{32a^2}{b^2}$ and $\frac{8a}{b}$ is

(vi) 1, 4, x , y are in continued proportion. $\therefore x = \dots, y = \dots$

(vii) x , 6, y and 54 are in continued proportion. $\therefore x = \dots, y = \dots$

2. If $a^2 + b^2$, $ab + bc$ and $b^2 + c^2$ are in continued proportion, prove that ' b ' is the mean proportional between a and c .

3. If $a + 2b + c$, $a - c$ and $a - 2b + c$ are in continued proportion, prove that ' b ' is the mean proportional between a and c .

4. If $b - c = \frac{2b}{a}$ and $a - b = 2$, prove that ' b ' is the mean proportional between a and c .

5. What number must be subtracted from each of the numbers 11, 15 and 21 so that the remainders will be in continued proportion ?

6. Find three numbers in continued proportion such that their sum is 13 and the sum of their squares is 91.

7. Three numbers are in continued proportion, of which the middle one is 16 and sum of the other two is 130. Find them.

8. Five numbers are in continued proportion such that the product of the first and the fifth is 64 and the sum of the second and the fourth is 20. Find them.

9. If a, b, c are in continued proportion, prove that

$$a^2 + b^2 + c^2 = (a + b + c)(a - b + c).$$

(Hint : $a^2 + b^2 + c^2 = a^2 + 2b^2 + c^2 - b^2$).

10. If a, b, c are in continued proportion, prove that $\sqrt{ab} + \sqrt{bc}$ is the mean proportional between $a + b$ and $b + c$.
11. Prove that the mean proportional between any two positive numbers is less than their average.

(Hint : Let $a > b > 0$. To prove that $\sqrt{ab} < \frac{1}{2}(a + b)$.
Note that $(a - b)^2 > 0$ and prove the result.)

§ 10.9 Some Problems on Equal Ratios :

Some results arising out of equal ratios can be proved easily by making each of the equal ratios equal to some number, say k .

Thus, if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots = k$, then we shall have $a = bk$, $c = dk$, $e = fk$, and so on.

The following examples will illustrate the method.

Example 10 : If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a + 3c}{b + 3d} = \frac{7a - 5c}{7b - 5d}$.

Let $\frac{a}{b} = \frac{c}{d} = k$, then $a = bk$, $c = dk$.

Substituting these values of 'a' and 'c', in the statement to be proved, we have

$$\text{L. H. S.} = \frac{a + 3c}{b + 3d} = \frac{bk + 3dk}{b + 3d} = \frac{k(b + 3d)}{b + 3d} = k \quad \dots (1)$$

$$\text{R. H. S.} = \frac{7a - 5c}{7b - 5d} = \frac{7bk - 5dk}{7b - 5d} = \frac{k(7b - 5d)}{7b - 5d} = k \quad \dots (2)$$

From (1) and (2), $\frac{a + 3c}{b + 3d} = \frac{7a - 5c}{7b - 5d}$ (each being 'k')

Example 11: If a, b, c, d are in continued proportion prove that—
 $a : d = (a^3 - b^2c + c^3) : (abc - bd^2 + d^3)$.

Since, a, b, c, d are in continued proportion, we have $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = k$ (say)

$\therefore c = dk, b = dk^2, a = dk^3$.

Substituting these values of a, b, c in terms of d and k , we have,

$$\frac{a}{d} = \frac{dk^3}{d} = k^3 \quad \dots (1)$$

$$\begin{aligned} \frac{a^3 - b^2c + c^3}{abc - bd^2 + d^3} &= \frac{d^3k^9 - d^3k^5 + d^3k^3}{d^3k^6 - d^3k^2 + d^3} \\ &= \frac{d^3k^3(k^6 - k^2 + 1)}{d^3(k^6 - k^2 + 1)} \\ &= k^3 \quad \dots (2) \end{aligned}$$

From (1) and (2), $\frac{a}{d} = \frac{a^3 - b^2c + c^3}{abc - bd^2 + d^3}$.

EXERCISE 10.6

1. If a, b, c and d are in continued proportion prove that—

$$(i) \frac{2a+5c}{2b+5d} = \frac{5a-3c}{5b-3d} \quad (ii) \frac{11a^2+9ac}{11b^2+9bd} = \frac{a^2+3ac}{b^2+3bd}$$

$$(iii) \frac{a^2+b^2}{ac+bd} = \frac{a^2-b^2}{ac-bd} \quad (iv) \frac{a^2+ab+b^2}{a^2-ab+b^2} = \frac{c^2+cd+d^2}{c^2-cd+d^2}$$

$$(v) (a-b) - (c-d) = \frac{(a-b)(a-c)}{a} \quad (vi) \frac{a}{d} = \frac{(a+b)(a-c)}{(b-d)(c+d)}$$

$$(vii) \frac{a}{b} = \sqrt{\frac{2a^2-5c^2}{2b^2-5d^2}} \quad (viii) \left(\frac{5a+3c}{5b+3d} \right)^2 = \frac{4a^2-9c^2}{4b^2-9d^2}$$

2. If a, b, c are in continued proportion, prove that

$$(i) \frac{a}{a+2b} = \frac{a-2b}{a-4c} \quad (ii) (a^2+b^2)(b^2+c^2) = (ab+bc)^2$$

$$(iii) \frac{b}{b+c} = \frac{a-b}{a-c} \quad (iv) (ab+bc+ca)^2 = abc(a+b+c)^2$$

3. If a, b, c, d are in continued proportion, prove that—

$$(i) \frac{b+c}{c-a} = \frac{d}{d-c} \quad (ii) (a-d)^2 = (b-c)^2 + (c-a)^2 + (d-b)^2$$

$$(iii) \frac{a+b}{b+c} = \frac{b+c}{c+d} \quad (iv) (ab+bc+cd)^2 = (a^2+b^2+c^2)(b^2+c^2+d^2)$$

§ 10.10 Theorem on Equal Ratios :

Let us consider the equivalent ratios, $\frac{4}{6}, \frac{8}{12}, \frac{2}{3}, \frac{10}{15}$.

Let us obtain a new ratio of the sums of the antecedents and the consequents of these ratios.

The new ratio = $\frac{4+8+2+10}{6+12+3+15} = \frac{24}{36}$. This new ratio is also equal to

each of the given ratios.

Let us now multiply both terms of each of the above ratios by non-zero numbers and form a ratio of the sums of those products. Thus

$$\begin{aligned} \text{The ratio} &= \frac{3(4) + 2(8) + (-5)(2) + \left(-\frac{1}{5}\right)(10)}{3(6) + 2(12) + (-5)(3) + \left(-\frac{1}{5}\right)(15)} \\ &= \frac{12 + 16 - 10 - 2}{18 + 24 - 15 - 3} = \frac{16}{24} \end{aligned}$$

This new ratio is also equal to each of the given ratios. The above result is generalised in the following theorem.

Theorem : If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots$, and if $l, m, n, \dots$ are non-zero numbers, such that $lb + md + nf + \dots \neq 0$, then each of the ratios

$$= \frac{la + mc + ne + \dots}{lb + md + nf + \dots}$$

Proof : Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots = k$

Then $a = bk, c = dk, e = fk, \dots$

$$\begin{aligned} \therefore \frac{la + mc + ne + \dots}{lb + md + nf + \dots} &= \frac{lbk + mdk + nfk + \dots}{lb + md + nf + \dots} \\ &= \frac{k(lb + md + nf + \dots)}{(lb + md + nf + \dots)} = k \\ \therefore \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots = \frac{la + mc + ne + \dots}{lb + md + nf + \dots} &= k. \end{aligned}$$

In problems on equivalent ratios, generally a hint can be taken from the problem itself. The following examples will make the point clear. Study them carefully.

Example 12 : If $\frac{b+c-a}{x} = \frac{c+a-b}{y} = \frac{a+b-c}{z}$, prove that

$$\frac{a}{y+z} = \frac{b}{z+x} = \frac{c}{x+y}.$$

Here we note that the denominators in the result to be proved are $y+z$, $z+x$ and $x+y$ respectively. So, we use the theorem on equivalent ratios so as to get the required denominators.

If each of the given ratios is equal to ' k ' (say for convenience)

$$\text{then } k = \frac{(b+c-a) + (c+a-b)}{x+y} = \frac{2c}{x+y} \dots \dots \dots (1)$$

$$\text{Similarly, } k = \frac{(c+a-b) + (a+b-c)}{y+z} = \frac{2a}{y+z} \dots \dots \dots (2)$$

$$\text{and } k = \frac{(a+b-c) + (b+c-a)}{z+x} = \frac{2b}{z+x} \dots \dots \dots (3)$$

$\therefore$ From (1), (2) and (3), we have,

$$\frac{2c}{x+y} = \frac{2a}{y+z} = \frac{2b}{z+x} \dots \dots \dots (4)$$

Multiplying each ratio in (4) by $\frac{1}{2}$, we get, $\frac{a}{y+z} = \frac{b}{z+x} = \frac{c}{x+y}$.

Example 13 : If $\frac{x}{a+2b+2c} = \frac{y}{b+2c+2a} = \frac{z}{c+2a+2b}$ prove

$$\text{that each ratio} = \frac{x+y-z}{a+b+3c}.$$

Here, the numerator in $\frac{x+y-z}{a+b+3c}$ gives us the clue as to how we must proceed.

Let each of the given ratios be, for convenience equal to k . Then by the theorem on equal ratios, we have

$$\begin{aligned} k &= \frac{x+y-z}{(a+2b+2c) + (b+2c+2a) - (c+2a+2b)} \\ &= \frac{x+y-z}{a+b+3c}. \end{aligned}$$

$$\therefore \frac{x}{a+2b+2c} = \frac{y}{b+2c+2a} = \frac{z}{c+2a+2b} = \frac{x+y-z}{a+b+3c}.$$

Example 14 : If $\frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b}$ and $a+b+c \neq 0$,

then prove that each ratio is equal to $\frac{1}{2}$.

For convenience, let us say, each of the given ratios is equal to 'k'.

Then, by the theorem on equal ratios, we have,

$$k = \frac{a+b+c}{(b+c) + (c+a) + (a+b)} = \frac{a+b+c}{2(a+b+c)} = \frac{1}{2}.$$

(Since $a+b+c \neq 0$).

EXERCISE 10.7

1. Fill in the blanks.

(i) $\frac{a}{7} = \frac{b}{3} = \frac{a-2b}{\dots\dots} = \frac{3a+5b}{\dots\dots}$.

(ii) $\frac{x}{3} = \frac{y}{4} = \frac{z}{5} = \frac{2x+3y}{\dots\dots} = \frac{4x-z}{\dots\dots} = \frac{\dots\dots}{3+4+5}$.

(iii) $\frac{x}{2} = \frac{y}{7} = \frac{z}{3} = \frac{5y-3z-4x}{\dots\dots}$

(iv) $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{2a+3c-5e}{\dots\dots} = \frac{\dots\dots}{5b+7d-11f}$.

(v) $\frac{x}{b+c} = \frac{y}{a-b} = \frac{\dots\dots}{a+c}$.

2. If $\frac{y+z}{a} = \frac{z+x}{b} = \frac{x+y}{c}$, then prove that $\frac{x}{b+c-a} = \frac{y}{c+a-b} = \frac{z}{a+b-c}$.

3. If $\frac{a}{b+c-a} = \frac{b}{c+a-b} = \frac{c}{a+b-c}$ and $a+b+c \neq 0$, then prove that each ratio is equal to 1.

4. If $\frac{ax+by}{x+y} = \frac{bx+az}{x+z} = \frac{ay+bz}{y+z}$ and $x+y+z \neq 0$,
then prove that each ratio $= \frac{a+b}{2}$.

5. If $\frac{a}{y+z-x} = \frac{b}{z+x-y} = \frac{c}{x+y-z}$, then prove that $\frac{x}{b+c} = \frac{y}{c+a} = \frac{z}{a+b}$.

6. If $\frac{y+z-x}{a} = \frac{z+x-y}{b} = \frac{x+y-z}{c}$, then prove that $\frac{x+y+z}{a+b+c} = \frac{x^2+y^2+z^2}{ay+bz+cx}$.

7. If $\frac{x}{a+2b+c} = \frac{y}{a-c} = \frac{z}{a-2b+c}$, then prove that
 $\frac{x+2y+z}{a} = \frac{x-z}{b} = \frac{x-2y+z}{c}$.

8. If $\frac{2x-3y}{3z+y} = \frac{z-y}{z-x} = \frac{x+3z}{2y-3x}$, then prove that each ratio is equal to $\frac{x}{y}$.

9. If $\frac{by+cz}{b^2+c^2} = \frac{cz+ax}{c^2+a^2} = \frac{ax+by}{a^2+b^2}$, then prove that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

10. Solve. $\frac{12x^2+18x+42}{18x^2+12x+58} = \frac{2x+3}{3x+2}$.

Chapter 11 : FUNCTION

§ 11.1 Function :

You know what is meant by one-to-one correspondence between two sets. When considering a congruence of two triangles we first of all observe a one-to-one correspondence between the vertices of the two triangles that might be a congruence. You are well aware of the one-to-one correspondence between the set of real numbers and the set of points on a line. We are now going to discuss a new concept called **function**. One-to-one correspondence is a special as well as a simple type of function.

Example 1 : We take two sets—one of some students and the other of some languages. In Fig. 11.1 an arrow is drawn connecting the name of a student with his mothertongue and this indicates the association of a student with his mothertongue.

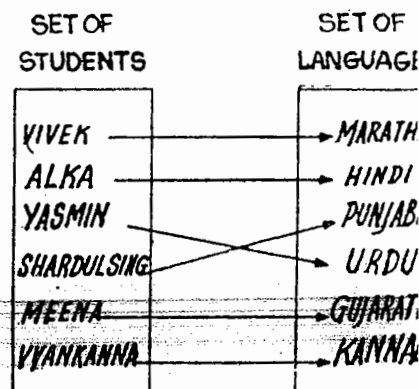


Fig. 11.1

The above diagram associates Vivek with his mothertongue which is Marathi; in the same way Alka is associated with Hindi, Yasmin with Urdu, Shardulsing with Punjabi, Meena with Gujarati and Vyankanna with Kannad. You will observe that only one arrow starts from the name of each student, as no one can have more than one mothertongue.

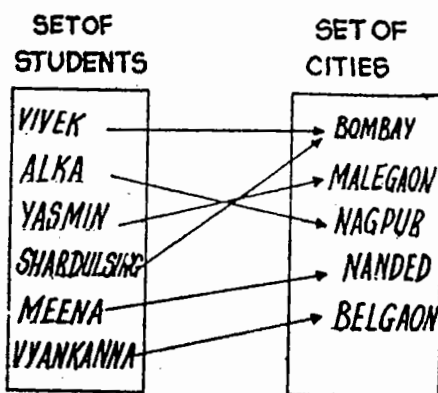


Fig. 11.2

Example 2 : In this example we take the same set of students as in example 1 above and for the second set we take the set of some cities. We associate each student with his place of birth. This association is indicated by means of arrows as shown in the adjoining diagram (Fig 11.2).

In this diagram also only one arrow starts from each student. Two arrows are seen to reach towards Bombay as it is the birth-place of both Vivek and Shardulsing. More than one arrow can point to the same city, but not more than one arrow can start from the same person as he cannot be born at two different places.

Example 3 : We again consider both the sets in example 1. But this time we are interested in the correspondence which associates with each student the languages he can speak.

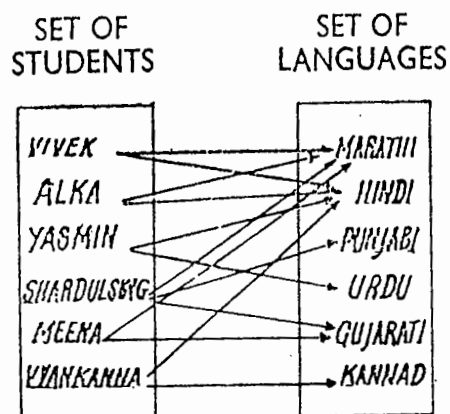


Fig. 11-3

Yasmin is associated with two languages, see the adjoining diagram. This means, she can speak the two languages, namely, Hindi and Urdu. Shardul Singh can speak three languages and hence three arrows start from his name.

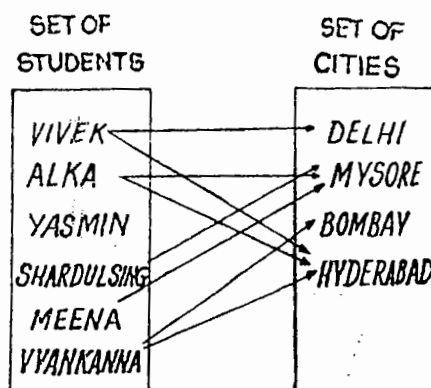


Fig. 11-4

Example 4 : Now for the first set we consider the same set of students as in the first three examples above and think of its correspondence with a set of some places. The correspondence exhibited in Fig. 11-4 shows the places each student has visited during the last three years on trips of his/her school.

The diagram above shows that Vivek has visited two cities, namely, Delhi and Hyderabad, while Yasmin has visited none.

In the above four examples we notice the following, regarding the correspondences exhibited in the above diagrams.

(1) In example one, at least one arrow starts from each of the elements in the set on the left and from any element of that set only one arrow is drawn. This means that with every element of the left-hand set is associated at least one element from the right-hand set and with any element of the left-hand set there is only one corresponding element of the right-hand set.

(2) In example 2, we meet the same situation with one important difference. One element from the right-hand set has two members of the left-hand set associated with it. Let us now observe how the correspondences in the third and the fourth examples are different from those in the first and the second examples.

(3) From some members of the left-hand set (in example 3) more than one arrow are directed towards the elements in the right-hand set. This means that some elements in the left-hand set are associated with more than one element in the

right-hand set. So, the important aspect peculiar to the correspondences of the first two examples, namely, 'any element of the first set is associated with **only one** element of the second set' is not fulfilled.

(4) In example 4 there is one element in the left-hand set which has no corresponding element in the right-hand set. So, in this case we cannot say that 'every element of the left-hand set is associated with **at least one** element of the right-hand set'.

We are interested in correspondences which satisfy the following two conditions.

(1) every element of the first set is associated with **at least one** element of the second set ; and

(2) for **any element** in the first set there is **only one** corresponding element of the second set.

We shall study only such correspondences in which both the above conditions are satisfied. Such correspondences are called 'functions'. The correspondences in the first two examples are functions, as they satisfy both these conditions ; but in case of the correspondence in example 3 above the second condition is violated and in the fourth example, the first condition is not satisfied. Hence, the correspondences in these examples are not functions.

EXERCISE 11.1

In the examples given below correspondences between two sets are shown. Decide which correspondences are functions. In case of those correspondences which are not functions, say which of the above conditions are not satisfied.

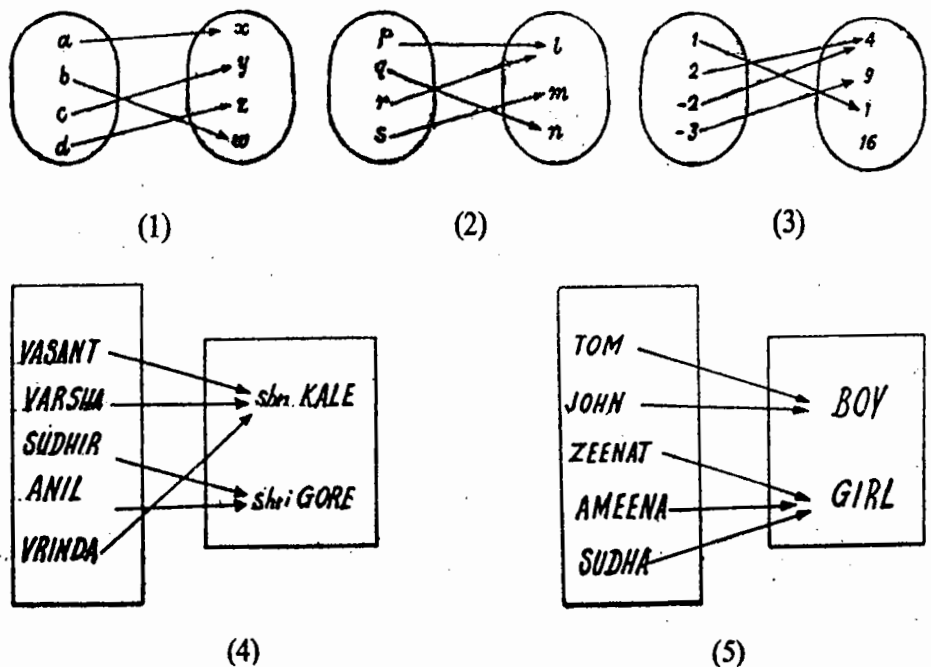
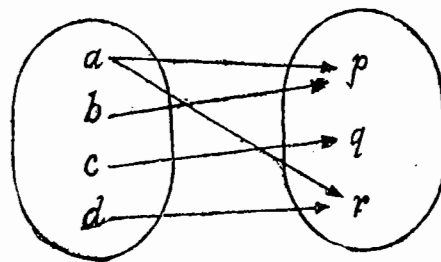
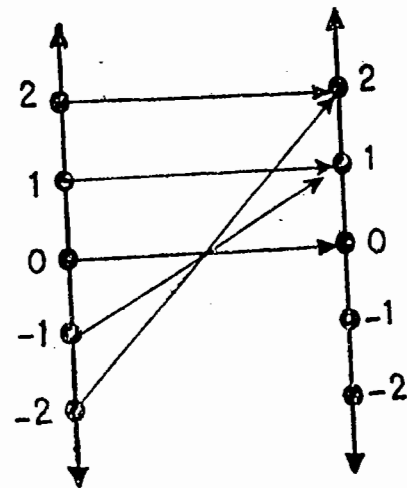


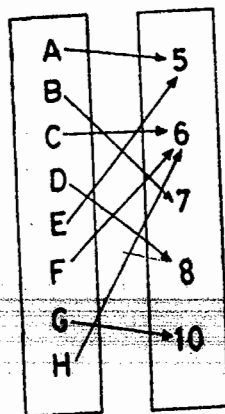
Fig. 11.5



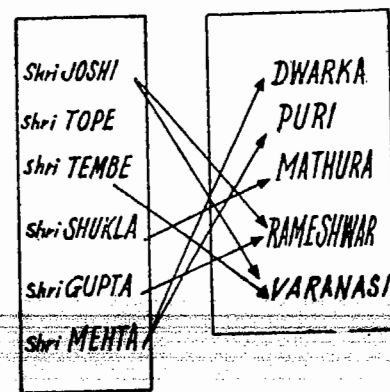
(6)



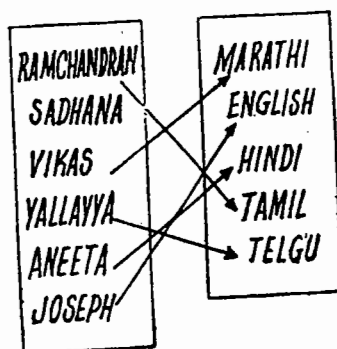
(7)



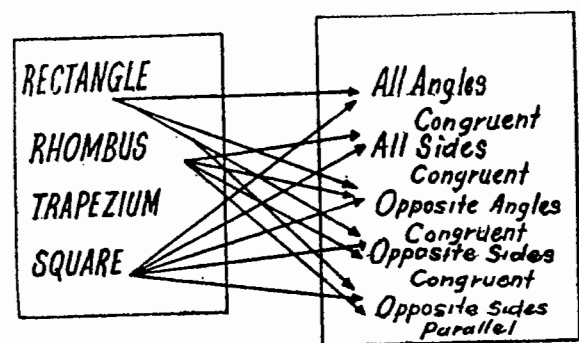
(8)



(9)



(10)



(11)

Fig. 11.5

If a correspondence between two sets A and B is a function then every element of A must be associated with at least one element of B and any element of A must be associated with only one element of B. This does not ensure that every element of the set B is associated with some member of the set A, or that every element of the set B is associated with only one element of the set A. (See examples 3 and 5 of Exercise 11.1). So a function is said to be 'from the set A on to the set B'.

If every element of the set B be associated with some element of the set A and any element of the set B is associated with only one element of the set A, then we have a special type of functions. However, we shall not discuss such distinctions in detail.

§ 11.2 Functional Notation:

Sometimes it is convenient to exhibit the correspondence between two sets A and B given by a function by means of a table of corresponding elements. This usually helps us in drawing graphs of functions.

When the sets are finite, we can write out all the pairs of corresponding elements in a table. The first element of such pairs is from the set A and the second element is the corresponding element from the set B. Let $A = \{-5, -3, -2, 2, 3, 5\}$ and $B = \{4, 9, 25\}$. A correspondence between these two sets is shown in the following table. It is a function.

Element in A	-5	-3	-2	2	3	5
Element in B	25	9	4	4	9	25

Many times a correspondence that is a function is given by some rule and this enables us to find the pairs of the corresponding elements. Therefore, sometimes such a rule is itself known as a 'function'. A correspondence which is a function is generally denoted by symbols like 'f', 'g', 'h'. If an element 'a' of the set A is associated with the element 'b' of the set B, by a function 'f', then 'b' is said to be the 'image' of 'a' under 'f'. Also 'b' is called the value of 'f' at 'a'. We write this as $f(a) = b$. For instance, if we consider the two sets A and B in the above example, then when $a = -5$, $b = 25$. So 25 is the value of 'f' at $a = -5$. When $a = 2$, then $b = 4$. So, 4 is the image of 2 under 'f'. We can express this as $f(-5) = 25$ and $f(2) = 4$.

If we denote any member of the set A by x and the corresponding element of the set B under 'f' by y, then we can as well write this as $y = f(x)$. The set A is known as the **domain** of the function 'f'.

We have said that many times a correspondence which is a function is given by a rule. Let us consider the same sets A and B as in the above example. You will observe that the elements in the set B are the squares of the corresponding elements of the set A. So if 'x' denotes an element of the set A and 'y' denotes an element of the set B corresponding to that element, then we have $y = x^2$. The correspondence given by the rule $y = x^2$ is a function because for any 'x' of A there is exactly one element of B associated with it. We say that ' $y = x^2$ ' is the rule which defines the said function.

We may as well show this correspondence by means of arrows starting from the elements of one set to the elements of the other as in the following diagram.

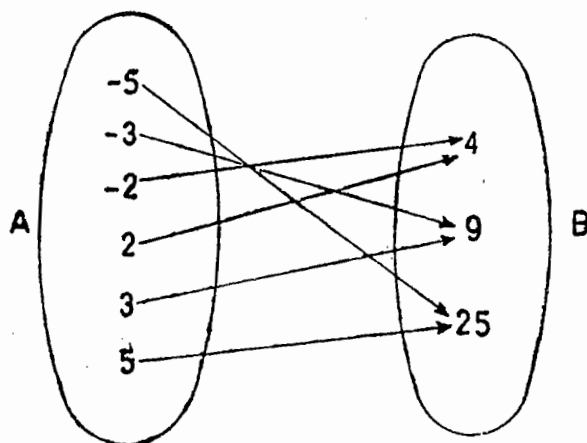


Fig. 11.6

B correspond to the same element of the set A. Such a correspondence as we have seen is not a function.

The above function may be fully defined by a set of ordered pairs, as shown below.

$$f = \{(-5, 25), (-3, 9), (-2, 4), (2, 4), (3, 9), (5, 25)\}.$$

When a function is exhibited as a set of ordered pairs, no two pairs of this set have first elements the same. For otherwise, it will mean that two elements of the set

Every element of the domain of a function must appear in at least one pair and any element of the domain-set cannot appear in more than one ordered pair.

In short, we can employ one of the following ways to define a function.

- (1) We may exhibit the correspondence by means of an arrow-diagram;
- (2) We can tabulate all the pairs of corresponding values of the elements in the two sets;
- (3) We may describe a function as a set of ordered pairs.
- (4) We can give a rule in words which helps us to associate the corresponding elements of the two sets;
- (5) We may define a function by a rule of correspondence denoted by a formula or an equation.

It must, however, be borne in mind that we may not be able to describe every function in all the above ways.

Example 5: A function 'f' is defined by an equation $y = 3x + 1$ and the domain of 'f' is the set $A = \{8, 6, 5, 4, -4, -3, -2, 0\}$. The function 'f' is then fully described by the following table—

x	8	6	5	4	-4	-3	-2	0
$y = 3x + 1$	25	19	16	13	-11	-8	-5	1

Some of the values of 'f' given by the above table may be written as $f(8) = 25$, $f(5) = 16$, $f(-4) = -11$, $f(0) = 1$.

Note that if the domain of 'f' is the set of all integers, we shall not be able to tabulate all the corresponding values.

Example 6: If function 'g' is given by the rule $y = x^2 - 3$ and 'x' is the element of the set $A = \{1, 2, -1, -2, 0\}$. Then, we can find the values of y, (i.e. of 'g') as follows—

$$\text{When } x = 1, \quad y = x^2 - 3 = (1)^2 - 3 = -2$$

$$\text{When } x = 2, \quad y = x^2 - 3 = (2)^2 - 3 = 1$$

$$\text{When } x = -1, \quad y = (-1)^2 - 3 = -2$$

$$\text{When } x = -2, \quad y = (-2)^2 - 3 = 1$$

$$\text{When } x = 0, \quad y = -3$$

With the help of the values obtained above, we can describe the function 'g' by the following set of ordered pairs—

$$\{(1, -2), (2, 1), (-1, -2), (-2, 1), (0, -3)\}.$$

The set of the first elements in the above set of ordered pairs is the set $A = \{1, 2, -1, -2, 0\}$ and it is the domain of 'g', and the set of the second elements is the set $B = \{-2, 1, -3\}$.

If a rule of correspondence of the form $y = f(x)$ defining a function is given, for an element x of the domain-set we can associate an element y given by the rule. We can form ordered pairs such as (x, y) . As x varies over the domain-set, we get different ordered pairs. We have seen that the set of all such ordered pairs describes the function. We, therefore, write $f = \{(x, y) | y = f(x)\}$.

Example 7: $h = \{(x, y) | y = x^2 + 2\}$ is a function.

The rule of correspondence in this function is $y = h(x) = x^2 + 2$.

As the domain of this function is the set of real numbers, it is not explicitly stated. We have the domain-set $A = \mathbb{R}$, here.

When $x = 0$, $y = 2$, and when the value of x is other than zero, x^2 is positive, and therefore, the value of $x^2 + 2$ becomes greater than 2. Hence, the set of images of x is the set $B = \{y | y \geq 2\}$.

Example 8: 'h' is a function whose domain is the set of real numbers and its rule of correspondence is $h(x) = 2x^2 - \frac{1}{2}x$. Find the values of $h(1)$, $h(2)$, $h(-3)$, $h(a)$ and $h(a+2)$.

$$h(1) = 2(1)^2 - \frac{1}{2}(1) = 2 - \frac{1}{2} = \frac{3}{2}$$

$$h(2) = 2(2)^2 - \frac{1}{2}(2) = 8 - 1 = 7$$

$$h(-3) = 2(-3)^2 - \frac{1}{2}(-3) = 18 + \frac{3}{2} = \frac{39}{2}$$

$$h(a) = 2(a)^2 - \frac{1}{2}(a) = 2a^2 - \frac{1}{2}a.$$

$$h(a+2) = 2(a+2)^2 - \frac{1}{2}(a+2) = 2a^2 + 8a + 8 - \frac{1}{2}a - 1 \\ = 2a^2 + \frac{15}{2}a + 7.$$

§ 11.3 Domain of a Function :

Strictly speaking it will be wrong to suppose that a function is fully described by stating a rule of correspondence. Firstly, we must ensure that to an element of set A the rule of correspondence does not associate more than one member—in fact it must associate exactly one member of the second set. So, we must invariably state

the domain of a given function. Suppose, 'g' is a function such that for every x in its domain, the corresponding element is x^2 . This statement does not define the function 'g' completely. Instead, if we mention that 'g' is a function such that for $-2 < x < 2$, $g(x) = x^2$, then the function is completely defined.

Suppose 'f' is a function, whose domain is $-2.5 < x < 8$ and $f(x) = x^2$. We see that the rule of correspondence of both the functions 'f' and 'g' is the same, but that their domains are different. In this case we will say that the functions 'f' and 'g' are different functions.

In general, even if the rule of correspondence of two functions is the same, they are different functions if their domains are different. For this very reason, the domain of a function must be clearly mentioned. So, when the domain of a function and the rule of correspondence are mentioned, then we say that the function is fully described.

Generally, however, when the domain of the function is not stated, it is understood that the domain is the set of real numbers. In case of some functions, their domain is determined from their rules of correspondence. For instance, consider a function 'f' such that $f(x) = \sqrt{9 - x^2}$. Then the domain of x is seen to be all real numbers such that $-3 \leq x \leq 3$. Because, if the value of x is greater than 3 or less than 3, the number under the radical sign will be negative, and for such values of x there is no real number which can denote $f(x)$. In such cases, we say that for these values of x , $f(x)$ does not exist and also that for such values 'f' is not defined.

If $g(x) = \frac{2}{x-1}$ is the rule of correspondence of a function 'g', then the domain of 'g' is the set of all real numbers excepting 1. For, when $x = 1$, as division by 0 is meaningless, there is no real number which is the value of the function $g(x)$.

§ 11.4 Linear and Constant Function :

'm' and 'h' are constant real numbers and $f(x) = mx + h$ is the rule of correspondence of the function 'f'. The equation $y = mx + h$ corresponds to this rule of correspondence. As you know, the graph of such an equation is a straight line. So, if a function 'f' is defined by the equation $f(x) = mx + h$, (m, h , constants), then 'f' is called a **linear function**. This means that if the rule of correspondence of a function is given by a first degree equation in two variables, the function is linear.

Example 9 : $f = \{(x, y) | 2y - 3x = 5\}$.

The equation $2y - 3x = 5$ is equivalent to the equation $2y = 3x + 5$, which is also equivalent to $y = \frac{3}{2}x + \frac{5}{2}$. It is of the form $y = mx + h$.

Therefore, $f(x) = \frac{3}{2}x + \frac{5}{2}$. Here $m = \frac{3}{2}$, $h = \frac{5}{2}$.

Hence, 'f' is a linear function.

If in the equation $g(x) = mx + h$, if m is 0, then $g(x) = h$. So, when $m = 0$, then, for every x in the domain of 'g', $g(x) = h$, where 'h' is a constant.

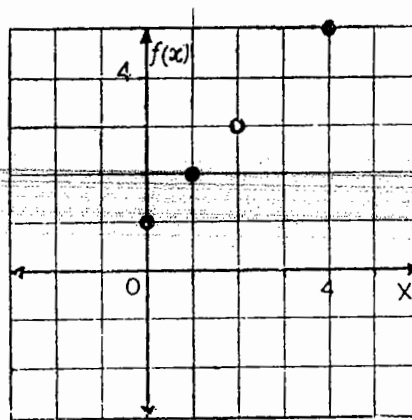
If 'g' be a function whose domain is the set of real numbers and the rule of correspondence is $g(x) = 3$, then some of the values of 'g' are tabulated below.

x	1	2	4.5	-6.3	$-\sqrt{7}$	$2-\sqrt{3}$
$g(x) = 3$	3	3	3	3	3	3

Such functions are called constant functions, for, for every element in the domain of the function, the value of the function is the same.

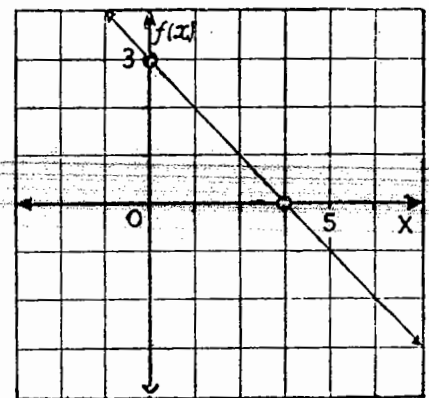
§ 11.5 Graphs of Functions :

We know the nature of the graphs of linear equations or equations of the first degree in two variables. You have also drawn graphs of some quadratic equations. A function is a set of ordered pairs as given by some equation; hence it can be demonstrated by a graph on the Cartesian plane. Fig. 11.7 shows graphs of some functions. Study them carefully.



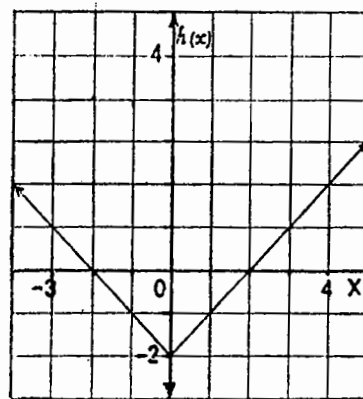
(1)

The graph of
 $f = \{ (0,1), (1,2), (2,3), (4,5) \}$



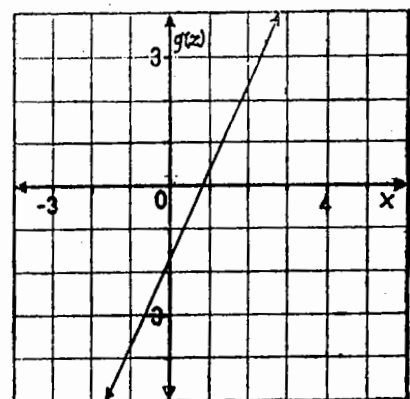
(2)

The graph of $f(x) = -(x-3)$



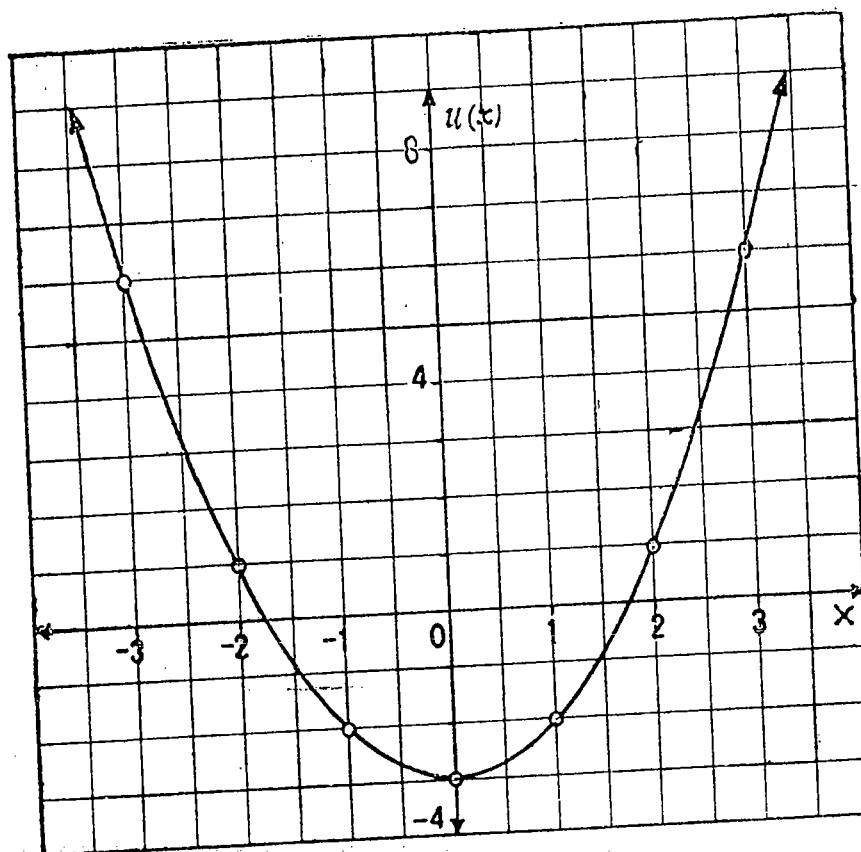
(3)

The graph of $h(x) = |x| - 2$



(4)

The graph of $g(x) = 2x - \sqrt{3}$



(5)

The graph of $u(x) = x^2 - 3$

Fig. 11.7

EXERCISE 11.2

State whether each of the following sets or the tables of corresponding values describe a function.

1. $\{(0, 1), (1, 3), (2, 7), (3, 1)\}$
2. $\{(0, 1), (2, 1), (4, 2), (6, 2)\}$
3. $\{(-1, 2), (-2, 4), (-3, 6), (-4, 8)\}$
4. $\{(1, -\frac{1}{2}), (-2, \frac{1}{2}), (3, -\frac{3}{2}), (-4, 2)\}$
5. $\{(1, 0), (3, 1), (7, 2), (11, 3)\}$

6.

x	1	1	2	2
y	0	2	4	6

7.

x	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$
y	1	-2	3	3

8.

x	-3	-5	-7	-9
y	4	4	4	4

9.

x	0	1	2	2	4	4
y	1	1	1	-1	1	-1

10. $\{(x, y) \mid y = |x - 2|, x \in \mathbb{R}\}$ 11. $\{(x, y) \mid x^2 + y + 3 = 0, x \in \mathbb{R}\}$
12. $\{(x, y) \mid y = \sqrt{x}, x \in \mathbb{R}, x \geq 0\}$
13. A function 'f' is defined by the rule $f(x) = x^2 + x$ and the domain of the function is the set of real numbers. Find—
- (i) $f(0)$ (ii) $f(\frac{1}{2})$ (iii) $f(1)$ (iv) $f(\sqrt{2})$
 (v) $f(-1)$ (vi) $f(-2)$ (vii) $f(-\frac{3}{2})$ (viii) $f(1 - \sqrt{3})$
14. Draw the graph of the function given in Ex. 10 above.
15. If the domain of the following functions is $\{1, 2, 3, 4, 5\}$ write them in the form of sets of ordered pairs.
- (i) $g(x) = x$ (ii) $g(x) = 6 - x$ (iii) $g(x) = 4$.
 (iv) $g(x) = -x$ (v) $g(x) = \sqrt{x}$ (vi) $g(x) = 2^{-x}$.
16. Draw graphs of the functions in Ex. 15 above.
17. If the domain of the following functions is the set of real numbers, say in each case, for what values of x the functions are not defined.
- (i) $f(x) = \sqrt{4 - x^2}$ (ii) $g(x) = \frac{x + 2}{x - 4}$ (iii) $h(x) = \frac{x^2 - 4}{x + 2}$
 (iv) $u(x) = 2 - x^2$ (v) $v(x) = \frac{x^2 - 6x - 7}{x^2 + 6x - 7}$
18. If $F(x) = 2^x$, find $F(-2)$, $F(1)$, $F(-\frac{1}{2})$, $F(\frac{1}{2})$.
19. (a) If $f(x) = (-1)^x$ and if the domain of 'f' is the set $\{1, 2, 3, 4, 5\}$, make a table of corresponding values in which the ordered pairs are of the form $(x, 1)$.
 (b) If $g(x) = 2^{-x}$ where the domain of 'g' is the set $\{-1, 0, 1\}$, write the set of ordered pairs of the form $(x, \frac{1}{2})$ in 'g'.
 (c) If $h(x) = 4$ and if the domain of 'h' is the set of natural numbers, write the set of ordered pairs of the form $(x, 2)$ in 'h'.
20. The domain of 'f' is $-3 \leq x \leq 3$ and $f(x) = 2|x|$. Find—
 (i) $f(0)$ (ii) $f(\frac{1}{2})$ (iii) $f(1)$ (iv) $f(-\frac{3}{2})$ (v) $f(-1)$ (vi) $f(-4)$.
 Draw the graph of $f(x)$.

Chapter 12 : VARIATION

§ 12.1 Constants and Variables :

We are familiar with the formula $A = \pi r^2$ which gives us the area of a circle, when its radius is known, and vice versa. Here we note that the area of a circle depends on the radius of the circle. As we go on changing the radius of the circle its area is also changed accordingly. Hence we say that 'A' and 'r' are variables. In this formula π is a constant. Though 'r' and 'A' change, π remains unaffected ; i.e. it does not depend on 'r'. In an open sentence or in a formula the symbols which take a fixed value even though other symbols take different values, are said to be constants.

§ 12.2 Direct Variation :

Observe the following table carefully and answer the questions given below.

No. of Books ..	2	3	8	10	15	x	?	a
Price in Rs. . .	10	15	40	50	75	60	y	b

- (1) From the table given above, find the value of ' x '.
- (2) How many books can be had for Rs. y ?
- (3) Form an equation connecting ' a ' and ' b '.

It is clear from the table that as we go on changing the number of books, the total price of the books also undergo a corresponding change. If the number of books is doubled the price of the books is also doubled. The ratio of 8 books to 3 books is the same as the ratio of the price of 8 books to the price of 3 books. That is

$$\frac{8}{3} = \frac{40}{15}.$$

Another important point that we can observe in the table is that the

ratio of numbers in each column is the same. It is 1 : 5.

It is exactly this second observation that has helped you to answer the questions asked above.

In general, we can say that if ' x ' represents the number of books and ' c ' their cost in rupees, then

$$\frac{x}{c} = \frac{1}{5}, \quad \frac{1}{5} \text{ is a constant.}$$

Fig. 12.1 shows the graph of the information given in the table on page 200.

Since the graph is a straight line through the origin we know that its equation should be of the form $y = mx$ where m represents the slope of the line. Find the slope of the graph that is drawn in the adjoining figure

and compare it with the ratio $\frac{c}{x}$.

What do you notice ?

Let us consider another example.

If 's' represents the side of an equilateral triangle, its perimeter 'p' is given by the formula $p = 3s$. Copy the following table in your note-book and complete it.

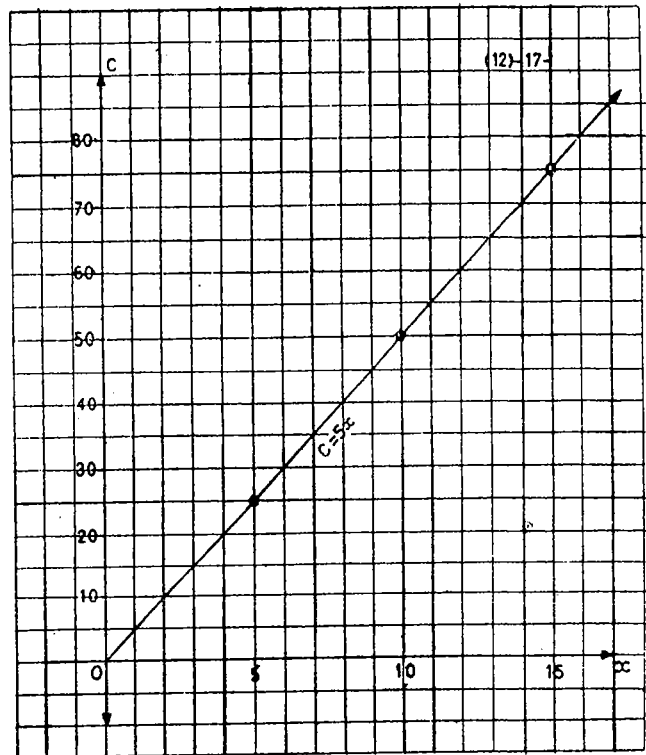


Fig. 12.1

s	1	2	3	4	5	6	7	8	9
$p = 3s$									

Using the table you have completed in your note-book, draw the graph. Compare this graph with the graph in fig. 12.1. What similarity do you find in both the graphs ?

If 's' is changed, how is 'p' affected ?

The information given in the above example can be stated in one of the following ways.

- (1) The perimeter of an equilateral triangle is proportional to the side of the triangle;
- (2) 'p' varies as 's'. In symbols we write this as $p \propto s$.
- (3) $p = ks$, where p and s are variables and k is a non-zero constant.
- (4) $\frac{p}{s} = k$, where p, s are variables and k is a constant.

The four statements given above mean the same thing. Let us consider another example.

The distance (s) through which a heavy body falls freely from rest varies as the square of the time (t) taken. The following table gives the corresponding values for ' t ' and ' s '—

t	1	2	3	4	5	6
$s = t^2$	4.8	19.2	43.2	76.8	120.0	172.8

Let us find the ratios of s and t for the corresponding values of s and t .

$$\frac{4.8}{1} = 4.8, \quad \frac{19.2}{2} = 9.6, \quad \frac{76.8}{4} = 19.2.$$

It is evident that the ratio $\frac{s}{t}$ is not constant and hence we can conclude that ' s ' does not vary as ' t '. This information is graphically represented in Fig. 12.2. Compare this graph with the graph shown in Fig. 12.1.

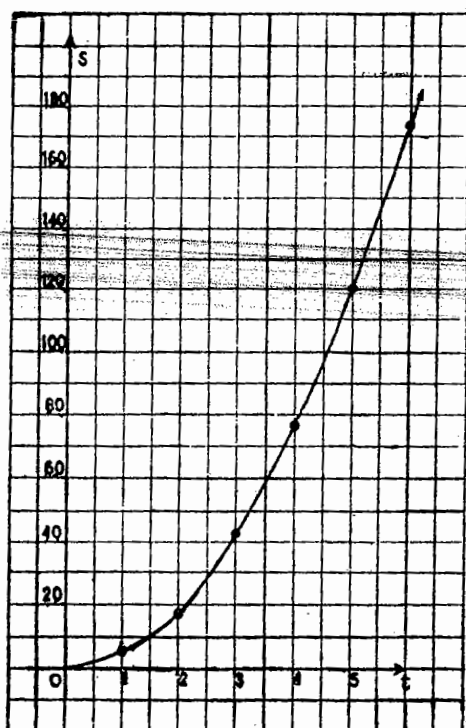


Fig. 12.2

Study the following table which has been prepared from the above table by replacing ' t ' by ' t^2 '.

t^2	1	4	9	16	25	36
s	4.8	19.2	43.2	76.8	120.0	172.8

From your table, find the ratio $\frac{s}{t^2}$. What do you notice?

The graph drawn from the table is shown in Fig. 12.3, and it is a straight line through the origin. Hence we can conclude that 's' varies as t^2 , or $s = kt^2$ where 'k' is a constant.

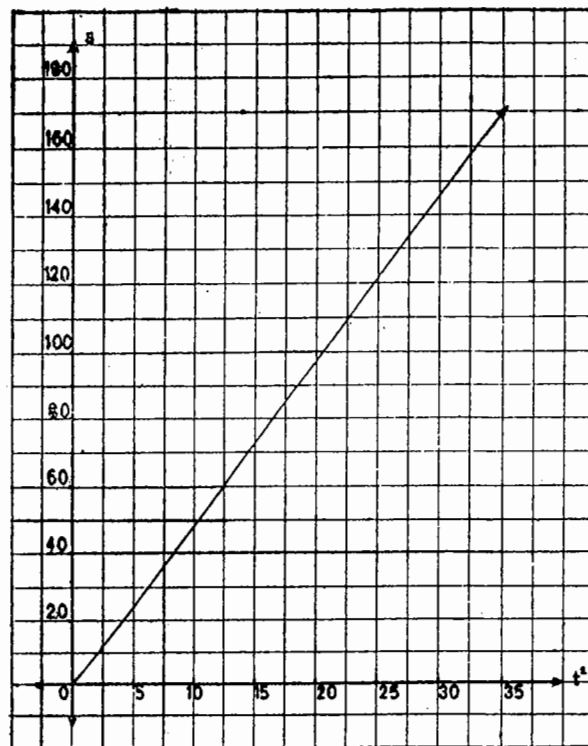


Fig. 12.3

Example 1 : If y varies as x and $y = 40$ when $x = 16$, find the value of y when $x = 24$.

$$y \propto x \quad \therefore y = kx \quad (\text{where 'k' is a constant}) \quad \dots (1)$$

It is given that $y = 40$ when $x = 16$.

$$\therefore 40 = 16k \quad \therefore k = 2.5$$

$$\therefore y = 2.5x \quad [\text{substituting the value of } k \text{ in (1)}].$$

$y = 2.5x$ is the equation of variation.

$$\therefore \text{when } x = 24, \quad y = 2.5 \times 24 = 60.$$

Example 2 : The circumference of a circle varies directly as its diameter. The circumference of a circle with diameter 10.5 cm is 33 cm. Find the constant of variation. Also find the diameter of the circle whose circumference is 88 cm.

Let the circumference be 'c' cm and let the diameter be 'd' cm. Then

$$c \propto d. \quad \therefore c = kd \quad (\text{where } k \text{ is a constant}) \quad \dots (1)$$

It is given that when $d = 10.5$; $c = 33$, $\therefore 33 = 10.5k$

$$\therefore k = \frac{33}{10.5}, \text{ which is the constant of variation.}$$

Substituting this value of 'k' in (1), we get,

$$c = \frac{33}{10.5}d, \text{ as the equation of variation.}$$

$$\therefore \text{when } c = 88, \text{ we have, } \therefore 88 = \frac{33}{10.5}d$$

$$\therefore d = 28$$

$\therefore$ the diameter of a circle whose circumference is 88 cm is 28 cm.

In general, a problem on variation has three stages. Stage 1 states the general law of variation. As in the above example we have, 'the circumference of a circle varies as its diameter'. From the information given in stage 2, we can calculate the constant of variation and substituting this value of the constant in the general law in stage 1, we arrive at the equation of variation. In the third stage, the equation obtained in stage 2 can be used to find the required answer.

EXERCISE 12.1

- In the following sets of ordered pairs the ratio $y : x$ is constant. Fill in the blanks.
 - $\{ (2, 4), (3, 6), (-2.5, \dots), (\dots, \frac{8}{3}) \}$.
 - $\{ (2, 5), (4, 10), (\dots, 20), (2.4, \dots), (\dots, 7) \}$
 - $\{ (1, 0.3), (5, 1.5), (4, \dots), (\dots, 0.21) \}$
- Answer the following questions with reference to the equation $3y = 2x$.
 - Can you conclude from the above equation that $y \propto x$? Is your conclusion true? Justify.
 - Does $x \propto y$? If so, state the constant of variation.
 - Fill in the blanks in the following ordered pairs.
 $(-3, \dots), (\dots, -3), (6, \dots), (2.1, \dots), (\dots, 1.2)$.
- In each of the following assume that y is proportional to x and find the constant of variation. Write out the equation connecting x and y .
 - $x = 1, y = 1$
 - $x = 2, y = 4$
 - $x = -2, y = 3$
 - $x = 1, y = 0.5$
 - $x = 2, y = -5$
 - $x = 7, y = 22$.
- If $y \propto x$, complete the following table and write down the equation connecting x and y .

x	2	3	8		
y	6	9		33	5

- If $y \propto x$ and $y = 5$ when $x = 2$, find the equation of variation and complete the following table.

x	1		5	0	
y		10			15

- Write each of the following statements in the form $y \propto x$, stating the meanings of the variables used in each case.
 - The circumference of a circle is proportional to its radius.
 - The height of an equilateral triangle is proportional to its side.
 - The volume of a sphere varies as the cube of its radius.
 - The square of the time taken by a planet to revolve round the sun varies as the cube of its mean distance from the sun.

7. If $y \propto x$ and $x = 18$ when $y = 15$. Find the equation connecting x and y and use it to find—
 (i) the value of y when $x = 4.5$, (ii) the value of x when $y = 2.5$.
8. If $y \propto x^2$ and $y = 1.5$ when $x = 1.5$. Then (i) find the value of y when $x = 4$, and (ii) find the value of x when $y = \frac{8}{27}$.
9. The area of a circular region varies as the square of its radius. The area of a circular region with radius 3.5 cm is 38.5 sq. cm. Find the area of the circular region whose radius is 14 cm.
10. The distance (d) covered in (t) seconds by an object falling vertically downwards from rest, varies as t^2 . In 2 seconds the object falls down nearly 2 metres. Find the distance it would fall in 6 seconds.
11. The cost of a diamond varies as the square of its weight. A diamond weighing 10 decigrams costs Rs. 1600 . Find the cost of a diamond of the same kind weighing 7 decigrams.
12. The area of a circular region varies as the square of its radius. Prove that the sum of the areas of two circular regions having radii $1\frac{1}{2}$ cm and 2 cm is equal to the area of a circular region with radius 2.5 cm.
13. If $a \propto b$, prove that $a^n \propto b^n$, ($n \in \mathbb{N}$).
14. If $a \propto b$, prove that $(a^2 - b^2) \propto ab$.
15. If $(a + b) \propto (a - b)$, prove that $a^2 + b^2 \propto ab$.

§ 12.3 Inverse Variation :

A cyclist travels 12 km in 3 hours. This means that his average speed is 4 km per hour. If he goes at an average speed of 8 km per hour, he would require only $1\frac{1}{2}$ hours for the same journey. This means that if the speed is doubled, the time taken is halved. If the speed be 2 km per hour instead of 4 km per hour he would require 6 hours to go a distance of 12 km. That is, the time required would be doubled. This means that the speed is inversely proportional to the time. Let us tabulate the above information.

Distance in km.	Time in hrs. (t)	Speed in km. p. h. (v)
12	3	4
12	6	2
12	12	1
12	1.5	8
12	1	12

It is evident from the above table that the product of the corresponding values of 't' and 'v' is 12, a constant. We can express this relation in one of the following ways.

$$(1) tv = 12 ; \text{ or } (2) t = \frac{12}{v} ; \text{ or } (3) v = \frac{12}{t} .$$

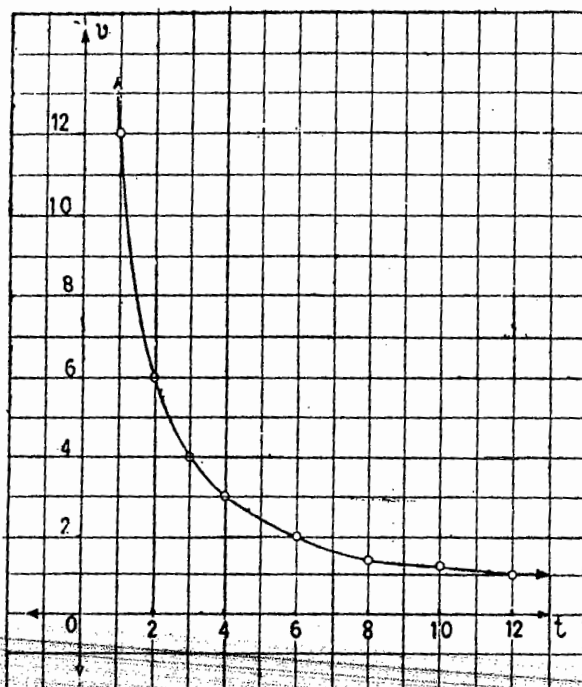


Fig. 12.4

We note that the product of a pair of corresponding values of 't' and 'v' must always be 12, which is a constant. In such a case we say that—

't' varies inversely as 'v'. Symbolically, we write this as $t \propto \frac{1}{v}$. The graph of the relation between 't' and 'v' is shown in the adjoining Fig. 12.4.

Let us consider another example.

24 men can do a piece of work in 60 days. If, instead of 24 men, we allow 12 men to complete the same work, they would require 120 days. 48 men can complete in 30 days, while 120 men will require only 12 days to complete it. If 'm' represents the number of men employed and 'd' the number of days required by them to complete the work, we can show the above information as in the chart below.

$\frac{m}{}$	$\frac{d}{}$	$\frac{m \times d}{}$
24	60	1440
12	120	1440
48	30	1440
120	12	1440

In all the above cases we note that the relation between 'm' and 'd' is given by

$$md = 1440 \text{ or } m = \frac{1440}{d} \text{ or } d = \frac{1440}{m} .$$

From the above discussion it is clear that —

- (1) 'm' is inversely proportional to 'd' or
- (2) 'd' is inversely proportional to 'm' or
- (3) m is directly proportional to $\frac{1}{d}$.

Example 3 : If 'y' varies inversely as 'x' and $y = 12$ when $x = 5$, find the value of y when $x = 10$.

Since y varies inversely as x , we have, $xy = k$,
(where k is a constant)(1)

But when $x = 5$, $y = 12$ $\therefore 5 \times 12 = k$
 $\therefore k = 60$

Hence, the equation of variation (1) becomes $xy = 60$.
 $\therefore$ when $x = 10$, we have $10y = 60$
 $\therefore y = 6$

Example 4 : For a given volume, the height 'h' of a cone varies inversely as the square of the radius 'r' of its base. Find the height of a cone of radius 4 cm if it is to have the same volume as a cone of radius 5 cm and of height 8 cm.

Here, 'h' varies inversely as the square of the radius 'r', of the base.
 $\therefore$ 'h' varies inversely as r^2 .

$\therefore hr^2 = k$ (a constant).....(1)

But when $r = 5$, $h = 8$ $\therefore 8 \times (5)^2 = k$ $\therefore k = 200$

$\therefore$ The equation of variation is $hr^2 = 200$.

$\therefore$ when $r = 4$, we have, $16h = 200$

$\therefore h = \frac{200}{16} = 12.5$

Hence, the height of the required cone is 12.5 cm.

§ 12.4 Hyperbola :

The graphs of direct variation and inverse variation should help you to see why these particular terms are used. First consider the graphs in Fig. 12.1 and Fig. 12.2 which are the graphs of direct variation. In this case if we move along the graph away from the origin, the absolute values of both x and y -co-ordinates of a point on the graph increase. In general, we may say, that in direct variation increasing absolute value of x is associated with corresponding increase in the absolute value of y .

Now, observe the graph of inverse variation shown in Fig. 12.4. The equation of the curve is $xy = 12$, i.e. it is of the form $xy = k$, (k is a constant). In this case if we move along the curve so that the absolute values of x -co-ordinates of points increase, then the values of the corresponding y -co-ordinates decrease.

Notice the form of the graph drawn in Fig. 12.4. It is a only part of the graph of the equation $xy = 12$ which is of the form $xy = k$, (k is a constant). The complete graph of the equation $xy = 12$ is shown in Fig. 12.5 on page 208. To draw this graph we have to extend the table prepared for Fig. 12.4. The extended table is given below.

x	1	2	3	4	6	12	-1	-2	-3	-4	-6	-12
$y = \frac{12}{x}$	12	6	4	3	2	1	-12	-6	-4	-3	-2	-1

The graph drawn from the table on page 207 is shown below.

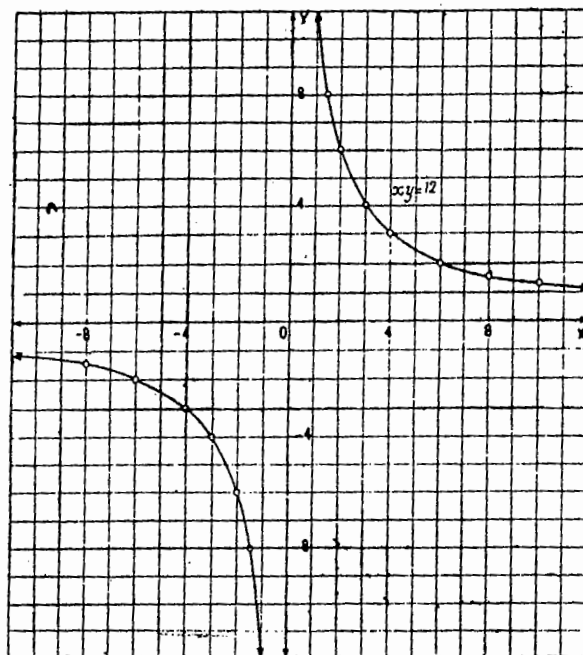


Fig. 12.5

Does the graph intersect the X-axis? Does it intersect the Y-axis? To answer these questions complete the following table.

x	10	100	1000	10000	1,000,000
$y = \frac{12}{x}$	1.2				

From the above table we observe that (1) As we increase the absolute value of x , the absolute value of y is diminished. Hence, the parts of the graph in both the first and the third quadrants approach the co-ordinate axes. However, note that they do not intersect the axes. (2) We have not taken the point whose abscissa is zero, while plotting the points. Why?

The curve representing the graph of the equation $xy = k$, (k is a constant), is called a **hyperbola**.

EXERCISE 12.2

1. Complete the following tables and say which of them represent relation of inverse variation.

(i)

x	3	12		0.5		120
y	8	2	24		96	

(ii)

p	45	120	105	180	
q	3	8	7		15

(iii)

r	12	4	15		
s	10	30		24	0.5

(iv)

u	72	108	6	2		
v	0.5	$\frac{1}{3}$			0.11	$\frac{1}{12}$

2. Fill in the blanks in the following ordered pairs with reference to the equation $xy = 30$.
 $(2, \dots), (-3, \dots), (5, \dots), (\dots, -6), (0.5, \dots), (\dots, -120)$.
3. If y varies inversely as x , find the constant of variation in each of the following and also write down the equations connecting x and y .
 (i) $x = 2, y = 3$ (ii) $x = \frac{1}{2}, y = 3$ (iii) $x = 0.5, y = 10$.
 (iv) $x = -2, y = 6$ (v) $x = 3.5, y = -2$ (vi) $x = -7, y = -3$.
4. Write the following statements in words.
 (i) $y \propto \frac{1}{x}$ (ii) $y \propto \frac{1}{x^2}$ (iii) $y \propto \frac{1}{\sqrt{x}}$.
5. If y varies inversely as x and $y = 30$ when $x = \frac{1}{3}$, then
 (i) Write down the equation connecting x and y ; (ii) Find the value of y when $x = 0.5$; (iii) Find the value of x when $y = 0.2$; (iv) What will be the effect on the value of y , if the value of x is doubled?
6. If $y \propto \frac{1}{x^2}$ and $y = 8$ when $x = 2$, then
 (i) find the value of y when $x = 4$;
 (ii) what is the effect on the value of y if the value of x is doubled?
7. y varies inversely as the cube-root of x and $y = 2$ when $x = \frac{1}{8}$; find the value of x when $y = 4$.
8. If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$, prove that $x \propto z$.
9. If $\left(\frac{1}{x} - \frac{1}{y}\right)$ varies inversely as $(x - y)$, then prove that $(x^2 + y^2)$ varies directly as xy .
10. In a simple electric current, the current ' i ' is inversely proportional to the resistance ' r ' and the current is 15 amperes when the resistance is 10 ohms. Find the current when the resistance is 25 ohms.
11. The illumination of a lamp on a screen (i) is inversely proportional to the square of the distance (d) of the screen from the lamp. When $d = 5, i = 1$, find the value of i when $d = 2.5$.
12. The attraction between two unlike magnetic poles varies inversely as the square of the distance between them. When the distance between such two poles is 12 cm there is an attraction equal to 2 gm weight. Find the attraction when the two poles are 8 cm apart.

§ 12.5 Joint Variation :

If 'l' and 'b' are the length and the breadth of a rectangular region, its area 'A' is given by the formula $A = l \times b$. We change the length of the rectangle and keep its breadth fixed. What effect will it have on its area? Now, keep its length fixed and go on changing its breadth and observe the variation in its area. The figures given below will help you to answer these questions.

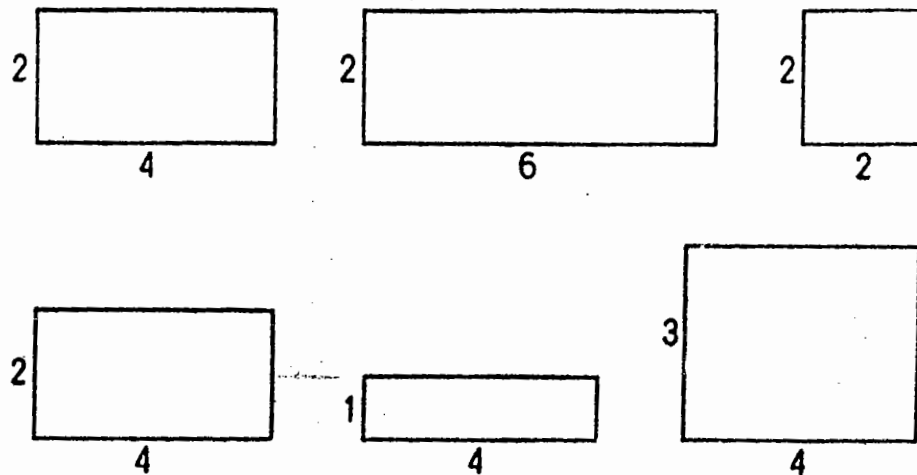


Fig. 12.6

If we change the length as well as the breadth of a rectangle, what will be the effect on its area? In such a case we say that the area of a rectangle **varies jointly** as its length and breadth.

When $l = 4$ and $b = 2$, the area is 8, and when $l = 6$ and $b = 3$, the area is 18. So, we can observe that the area changes directly as the product of the length and the breadth.

Let us consider one more example. Study the following chart carefully.

Principal, Rs.	100	200	200	400	500	250	x	a
Time, Years	1	1	3	4	2	5	y	b
Interest, Rs.	4	8	24	64	40	?	?	c

We know that in problems on simple interest, when the period and the rate of interest are fixed, the simple interest is directly proportional to the principal, and when the principal and the rate of interest are fixed, the simple interest is directly proportional to the period.

Find the product of the first two numbers in the first column, i.e. the product of numbers showing the principal and the period. Find the ratio of this product to the corresponding number in the third row of the first column showing interest. The

$$\text{ratio is } \frac{100 \times 1}{4} = 25.$$

Do this same thing to the numbers in the other columns. What do you notice ?

We note that the product of the principal and the period, is directly proportional to the interest.

Now fill in the blanks in the sixth and the seventh columns. Form an equation connecting a , b and c given in the eighth column.

From the above discussion it is clear that when the rate of interest is constant, the simple interest is directly proportional to the product of the principal and the time. This is a particular case of a general result which is given below.

If x varies as y when z is constant, and x varies as z when y is constant, then x varies as yz when both y and z vary.

We express the above statement symbolically as follows—

$$x \propto yz \text{ or } x = kyz, (k \text{ is a non-zero constant}) \text{ or } \frac{x}{yz} = k (k \text{ is a constant}).$$

One quantity is said to vary jointly as a number of other quantities if it varies directly as their product. Similarly, if ' a ' varies directly as ' b ' and inversely as

' c ', we have $a \propto \frac{b}{c}$, i.e. $a = \frac{kb}{c}$, (k is a constant, $k \neq 0$).

Example 5: Given that ' p ' varies jointly as ' a ' and ' b^2 ' and that $p = 18$ when $a = 9$ and $b = 15$, find ' b ' when $a = 81$ and $p = 32$.

' p ' varies jointly as ' a ' and ' b^2 '.

$$\therefore p = kab^2 (k \text{ is a non-zero constant}).$$

$$\text{When } a = 9 \text{ and } b = 15, \text{ we have } p = 18. \therefore 18 = k \times 9 \times (15)^2$$

$$\therefore k = \frac{2}{225}.$$

$$\text{Hence the equation of variation is } p = \frac{2}{225} ab^2.$$

Now, when $a = 81$ and $p = 32$, we get,

$$32 = \frac{2}{225} \times 81 \times b^2, \text{ which gives } b^2 = 32 \times \frac{225}{162} = \frac{400}{9}$$

$$\therefore b = \frac{20}{3} \text{ or } -\frac{20}{3}$$

EXERCISE 12-3

1. The area of a triangle varies jointly as its base and height. The area of a triangle having a base of 6 cm and the height of 4 cm is 12 sq. cm. Find the length and the height of a triangle whose base is 8 cm and area is 36 sq. cm.
2. The volume of a cylinder varies jointly as the square of the radius of its base and its height. The volume of the cylinder having a base of radius 7 cm and the height 10 cm is 1540 c.c. Find the radius of the base of the cylinder whose height is 14 cm and volume 396 c.c.

3. The resistance to an electric current varies directly as the length of the wire through which it flows and inversely as the square of the diameter of the wire. If the wire has diameter 2 mm and length 10 metres, the resistance is 64 ohms. Find the amount of resistance to the current which flows through a wire having diameter 9 mm and length 5 metres.
4. The volume of a right circular cone varies as its height when the area of its base is constant and as the area of its base when its height is constant. The volume of a cone having a base area of 36 sq. cm and height 7 cm is 84 c.c. Find the height of the cone whose volume is 168 c.c. and has a base of area 28 sq. cm.
5. The weight of a solid sphere varies as the density of its material when its diameter is constant and as the diameter when the density is constant. A sphere having density 2.1 and diameter 2 cm weighs 8.8 gm. Find the weight of a sphere whose diameter is 1 cm and density 6.3.
6. The area of a sector of a circle varies as the product of its radius and its arc. If the length of an arc is 12 cm and the radius of the sector is 8 cm, the area of the sector is 48 sq. cm. Find the area of a sector whose radius is 10 cm and the length of its arc is 17 cm.
7. The volume (v) of a gas varies directly as the absolute temperature (t) and inversely as the pressure (p). The volume of a certain gas at temperature 360°A and pressure 736 mm is 450 c.c. Find its volume at temperature 312°A and pressure 960 mm.
8. The time required by an object to slip down an inclined plane varies directly as the square of the distance and inversely as the height of the object from the horizontal plane. If the length of the inclined plane be 12 cm and the height be 6 cm, the time required for an object to slip down from the topmost end of the inclined plane to its downmost end is 2 seconds. Find the time required by an object to slip down the whole inclined plane of length 18 cm and height 1.5 cm.
9. The sum of the distances of an object and its image from a mirror varies directly as the product of those distances. The image of an object at a distance of 300 cm from a mirror appears to be 120 cm away from the mirror. An object is placed at a distance of 540 cm from a mirror. Find the distance from the mirror at which the image will appear.
10. The profit in running a bus service varies directly as the distance when the number of passengers is constant, and varies directly as the number of passengers in excess of a certain number, when the distance is constant. The gain in carrying 19 passengers over a distance of 19 km is Rs. 12, and the gain in carrying 28 passengers over a distance of 20 km is Rs. 30. Find how many passengers can be carried over without any loss?

§ 12.6 Other Forms of Variation :

Consider Fig. 12.7(1). We can divide it into two parts, of which one is a rectangular region and the other is a semi-circular region. The diameter of the semi-circular region is the length of the rectangle.

We observe here that the area of the semi-circular region will not be affected even though we increase or decrease the breadth of the rectangle. See Fig. 12.7(2). Let the area of the rectangular region be 'R' and the area of the semi-circular region be 'C'. Denote the area of the whole figure by 'A'.

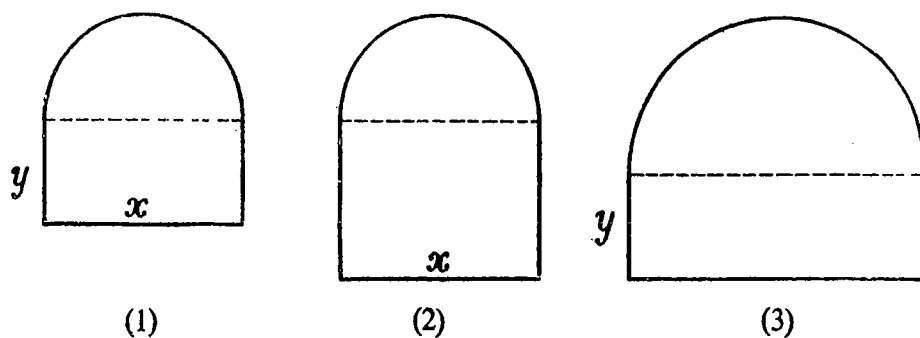


Fig. 12.7

Then, $A = R + C$.

Now, let the length and breadth of the rectangle be $2a$ and x units respectively.

$$\therefore R = 2a \times x = 2ax, \text{ and } C = \frac{1}{2} \pi a^2.$$

$$\therefore A = 2ax + \frac{1}{2} \pi a^2.$$

If we keep 'A' fixed and allow x to vary, then the value of A also will vary with x . But this does not mean that A varies as x . We know that A is the sum of two quantities of which one is fixed, (namely πa^2) and the other, namely $2ax$ varies as x .

Here, we say that 'A is partly constant and partly varies as x '.

Now, see Fig. 12.7 (3). In that figure, we have kept the breadth of the rectangle fixed and have allowed its length to vary. Let the length and the breadth of the rectangle be $2y$ and b respectively.

$$\text{Then, } R = 2y \times b = 2by \text{ and } C = \frac{1}{2} \pi y^2.$$

$$\therefore A = 2by + \frac{1}{2} \pi y^2.$$

Here, A is the sum of two quantities of which one varies as y and the other varies as the square of y .

In general, if x varies partly as y and partly as the square of y , we have, $x = ky + my^2$, (k and m are non-zero constants).

It should be carefully noted that we must use different constants of variation for y and y^2 .

Observe the following examples—

Example 6 : $x = y + z$ and $y \propto p$ and $z \propto p^2$. $x = 16$ when $p = 2$, and $x = 33$ when $p = 3$. Find x when $p = 5$.

We have, $y \propto p$ and $z \propto p^2$.

$\therefore y = mp, \quad z = np^2 \quad (m, n \text{ constants}).$

$\therefore x = y + z = mp + np^2$ (1)

If we substitute $p = 2, x = 16$ in (1), we get,

$$16 = 2m + 4n$$
 (2)

Similarly, substituting $p = 3, x = 33$ in (1), we get,

$$33 = 3m + 9n$$
 (3)

Solving (2) and (3), we get, $m = 2, n = 3$.

Substituting these values in (1), it becomes,

$$x = 2p + 3p^2$$
 (4)

Which is the equation of variation.

Now, when $p = 5$, we get from (4),

$$x = 2 \times 5 + 3 \times (5)^2$$

$$= 10 + 75$$

$$= 85.$$

Example 7: The expenses of a social function of a school are partly constant and partly vary as the number of pupils taking part in it. When 400 pupils participated, the expenses were Rs. 6 per pupil. If 40 pupils had been absent, each pupil would have to pay 50 paise extra. Find the total expenses of the social function when 500 pupils participate.

Let Rs. E be the total expenses of which Rs. E_1 is the part that is constant and let E_2 be the part that varies as the number of pupils participating, which we take to be ' x '.

$\therefore E_2 \propto x$. That is, $E_2 = mx$ (where m is constant).

$$\therefore E = E_1 + mx$$
 (1)

From the first condition in the problem, we have,

$$400 \times 6 = E_1 + 400m$$
 (2)

The second condition in the problem gives us

$$360 \times (6 + \frac{1}{2}) = E_1 + 360m$$
 (3)

Solving (2) and (3), we get, $E_1 = 1800$ and $m = \frac{3}{2}$.

$\therefore$ The equation of variation (1), becomes

$$E = 1800 + \frac{3}{2}x$$
 (4)

$\therefore$ when 500 pupils participate,

$$E = 1800 + \frac{3}{2}(500) = 1800 + 750 = 2550.$$

$\therefore$ when 500 pupils participate the total expenses of the social function would be Rs. 2550.

EXERCISE 12.4

1. $x = a + y$ and 'a' is constant while $y \propto z$. $x = 30$ when $z = 6$, and $x = 24$ when $z = 2$. Find x when $z = 10$.
2. y is equal to the sum of two quantities of which one is constant and the other varies as x . $y = 19$ when $x = \frac{1}{2}$ and $y = 47$ when $x = 4$. Find y when $x = -\frac{1}{4}$.
3. x is partly constant and partly varies as y^3 . $y = 2$ when $x = 19$ and $y = 8$, when $x = -44$. Find y when $x = -7$.
4. The expenses of a hostel are partly constant and partly vary as the number of inmates. The expenses are Rs. 4075 and Rs. 5875 respectively when the number of inmates are 30 and 45. Find the expenses of the hostel when there are 34 inmates.
5. The expenses of a function of a school are partly constant and partly vary as the number of participating students. 320 students taking part in the function would have to contribute Rs. 2 each. If 20 students had been unable to attend the function, the remaining students would have to pay 5 paise more each. Find the expenses of the function if only 240 students had participated.
6. The expenses of a trip are partly constant and partly vary as the number of days the trip lasts. A 20 days' trip costs Rs. 360 while a thirty days' trip costs Rs. 490. Find the cost of a trip that would last for 24 days.
7. A person has to pay income tax on the portion of his income that exceeds a certain amount. A person whose income is Rs. 7240 pays Rs. 224 as income-tax and a person having an income of Rs. 9460 has to pay Rs. 446 as income-tax. Find the income of the person who pays an income-tax of Rs. 315.
8. The cost of digging a well partly varies as its depth and partly varies as the square of its depth. The cost of digging a well 10 metres deep is Rs. 2000, while that of a well 15 metres deep is Rs. 4125. Find the cost of digging a well 8 metres deep.

ANSWERS

Exercise 1.1

- (1) $2 < x < 6$. (2) $-3 < x < 1$. (3) $1 < x < 3.5$ (4) $-4 < x < -2$.

Exercise 1.2.

- (1) $|x| = 2$. (2) $|x| > 2$. (3) $|x| > 3$. (4) $(x - 2)(x + 3) = 0$.
 (5) $(x + 5)(2x - 6) = 0$. (6) $(x - 4)(x - 5) = 0$. (7) $x \leq 3$. (8) $2x + 3 > 8$.

Exercise 1.3

- (1) $\{x | x > 7\}$. (2) $\{x | 1 < x\}$. (3) $\{x | x < 3\}$. (4) $\{x | x > -5\}$.
 (5) $\{x | x \leq 4\}$. (6) $\{x | x > -5\}$. (7) $\{x | x \geq -\frac{23}{6}\}$.
 (8) $\{x | -\frac{1}{2} < x\}$. (9) $\{x | x < 2\}$. (10) $\{x | x > 7.5\}$.
 (11) $\{x | x > 3\}$. (12) $\{x | x \leq -\frac{14}{3}\}$. (13) $\{x | x > 3 \text{ or } x < 3\}$.
 (14) $\{x | x \neq -\frac{2}{3}\}$. (15) $\{x | x > \frac{3}{4} \text{ or } x < \frac{7}{4}\}$. (16) $\{x | x \geq -42 \text{ or } x < 98\}$.
 (17) $\{x | 2 \leq x \leq 4\}$. (18) $\{x | 5 \geq x > 2\}$.
 (19) (i) 0, (ii) positive, (iii) 0, (iv) positive, (v) negative, (vi) negative.
 (20) (i) 0, (ii) positive, (iii) 0, (iv) positive, (v) negative, (vi) negative.
 (21) $\{x | x > 2 \text{ or } x < -2\}$. (22) $\{x | 1 < x < 4\}$.
 (23) $\{x | x \geq 0 \text{ or } x \leq -5\}$. (24) $\{x | x > 4 \text{ or } x < -4\}$.
 (25) $\{x | -4 < x < 2\}$. (26) $\{5, 6, 7, 8, \dots\}$. (27) $\{1, 2, 3, 4, 5, 6\}$. (28) 5, 4 cm.
 (29) negative. (30) positive. (31) negative.
 (32) (i) a ray, (ii) a ray, (iii) a ray, (iv) a segment.

Exercise 2.2

- (1) 12. As each of the three elements of the set A is associated with four elements each of the set, $3 \times 4 = 12$ is the number of pairs.
 (2) $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)\}$
 (3) 20.

(4) $\{ (a, 2), (a, 5), (b, 2), (b, 5), (c, 2), (c, 5), (d, 2), (d, 5) \}$.

(5) (i) $x = 2, y = -2$. (ii) $x = 4, y = -6$.

Exercise 2.3

(1) I, X-axis, IV, III, I, IV, IV, III, II, IV, I, X-axis, Y-axis, X-axis, Y-axis.

(2) (i) (0,0), (ii) (5,0), (iii) (6,10), (iv) (0, 5).

(3) A (2, 1), B (-2, 1), C (4, 4), D (4, -4), E (-2, 3), F (-2, -3),
G ($3\frac{1}{2}$, 2), H ($-3\frac{1}{2}$, 2), J ($-3\frac{1}{2}$, -4), K ($2\frac{1}{2}$, -4), L ($-4\frac{1}{2}$, $2\frac{1}{2}$),
M ($1\frac{1}{2}$, $3\frac{1}{2}$), N ($1\frac{1}{2}$, $-2\frac{1}{2}$), P ($-1\frac{1}{2}$, $-\frac{1}{2}$).

(4) (i) y-co-ordinate, (ii) the origin, (iii) Y-axis, (iv) second, (v) negative,
(vi) fourth.

Exercise 2.4

(1) Q (-3, 4), M (-7, -5), L (6, -5), Y (-3, 7).

Q - $x = -3, y = 4$; M - $x = -7, y = -5$;

L - $x = 6, y = -5$; Y - $x = -3, y = 7$.

(2) (i) $y = -3$; (ii) $x = a, (a \in \mathbb{R})$; (iii) (-2,0); (iv) $y = -8$.

Exercise 3.1

$y = x$.

x	0	-5	-10	3	8	-12	12	5
y = x	0	-5	-10	3	8	-12	12	5

$y = x + 6$.

x	0	-5	-10	3	8	-12	-18	6
y = x + 6	6	1	-4	9	14	-6	-12	12

$y = x - 8$.

x	0	-5	-10	3	8	-4	6	4
y = x - 8	-8	-13	-18	-5	0	-12	-2	-4

Exercise 3.2

- (1) (a) $y = -\frac{1}{2}x$, (b) $y = 2x$, (c) MN, $y = \frac{1}{2}x$, (d) $(2, -5)$ $(-4, 10)$, $(4, -10)$, $y = -\frac{5}{2}x$. (e) $y = \frac{3}{2}x$, $y = 3x$.
- (2) (a) $y = 2x$, (b) $y = -\frac{1}{2}x$, (c) $y = \frac{1}{2}x$.
- (3) $y = \frac{1}{2}x$, $y = 7x$, $y = -3x$, $y = -\frac{1}{2}x$.
- (4) $(1, -2)$, $(4, -8)$, $(-3, 6)$, $(-6, 12)$; $y = 2x$, $y = -2x$.
- (5) $(-1, 2)$, $(-4, 8)$, $(3, -6)$, $(6, -12)$.

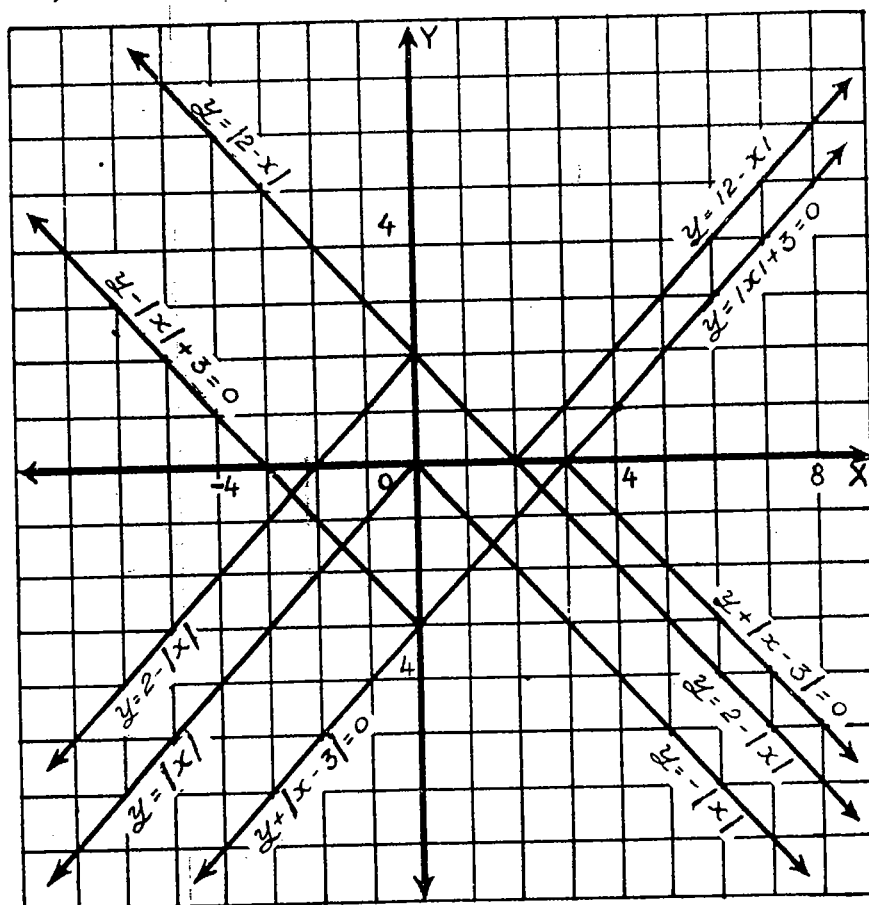
Exercise 3.3

- (1) (a) $y = 2x + 1$, (b) $(3, 7)$, (c) $(-3, -5)$, (d) $y = 2x + 1$.
- (2) $(0, h)$ $\left(-\frac{h}{2}, 0\right)$
- (3) The graphs are parallel to each other. In all the equations the coefficient of x is the same namely, 2.
- (4) $y = -2x + 5$, 5.

Exercise 3.4

- (1) $y = -x + 1$. (2) $y = -x + 7$. (3) $y = 2x + 8$.
- (4) $y = -\frac{1}{2}x + 6$. (5) $y = -\frac{3}{5}x + \frac{9}{5}$. (6) $y = \frac{4}{7}x + \frac{8}{7}$.
- (7) $-\frac{1}{2}$. (8) 3, 2. (9) $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$. (10) 2. (11) 5, $\frac{20}{3}$.
- (12) $(3, -2)$. (13) 2. (14) $(-6, -3)$. (15) $3x + 2y - 6 = 0$.
- (16) $A = 2$, $B = 3$, $C = 6$.

Exercise 3.5

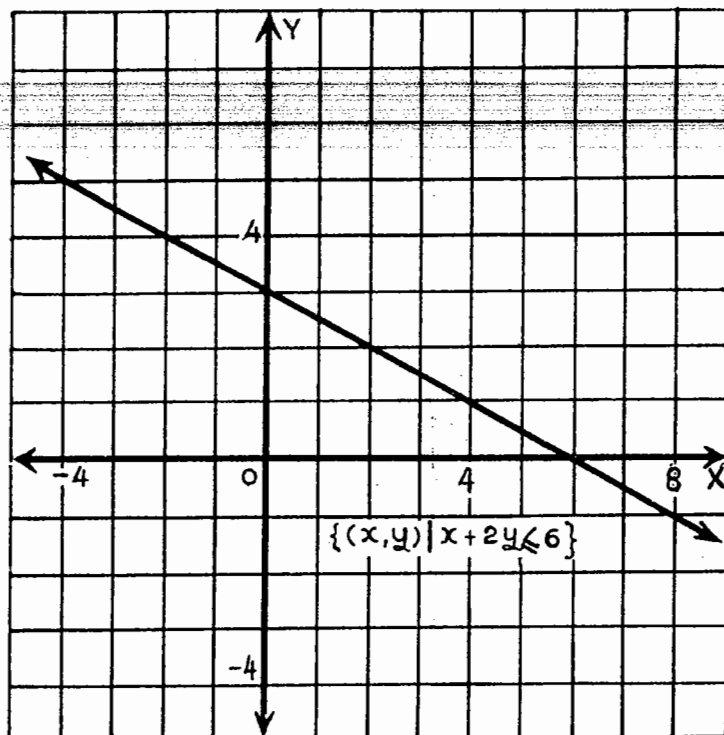


Exercise 3.6

(1)

Point	Co-ordinates	Value of $2x + y$	Relation with 2	Where does the point lie ?
A	$(-2, 6)$	2	$= 2$	On the line
C	$(4, 4)$	12	> 2	In H_2
D	$(-2, 3)$	-1	< 2	In H_1
E	$(-5, -3)$	-13	< 2	In H_1
G	$(3, -2)$	4	> 2	In H_2
H	$(7, -5)$	9	> 2	In H_2
K	$(-1, 2)$	0	< 2	In H_1
L	$(3, -4)$	2	$= 2$	On the line

(2)

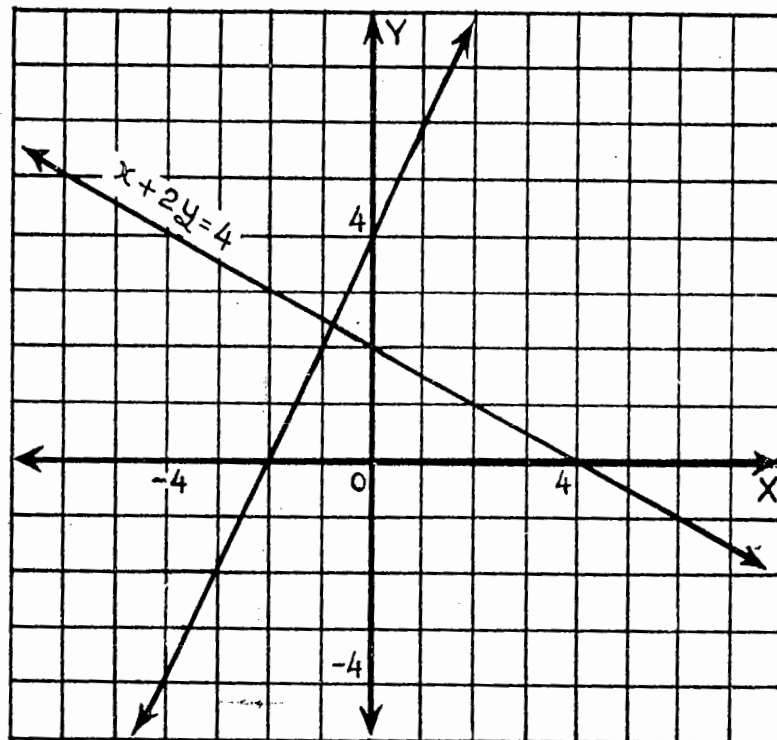


(3)

	(a)	(b)	(c)	(d)	(e)	(f)
Point	$(10, -15)$	$(8, 8)$	$(-5, -10)$	$(0, 4)$	$(0, 0)$	$(-15, 20)$
Set	M	N	M	L	N	L

(4) $\{(x, y) \mid 3x + 2y > 6\}$.

(5)



$$\{(x, y) \mid x + 2y \leq 4\} \cap \{(x, y) \mid y - 2x \leq 4\}$$

Exercise 4.1

- (1) $\{(-1, 1)\}$ (2) $\{(\frac{7}{2}, -\frac{1}{2})\}$ (3) $\{(3, 2)\}$ (4) $\{(2, -2)\}$ (5) $\{(-1, 1)\}$.

Exercise 4.2

- (1) $\{(2, 3)\}$ (2) $\{(1, -3)\}$ (3) $\{(8, 5)\}$ (4) $\{(-2, 4)\}$ (5) $\{(3, 4)\}$ (6) $\{(7, 2)\}$

Exercise 4.3

- (1) $\{(2, -1)\}$, (2) $\{(2, 3)\}$ (3) $\{(3, 1)\}$, (4) $\{(3, 1)\}$, (5) $\{(3, -1)\}$.

Exercise 4.4

- (1) $\{(4, 3)\}$, (2) $\{(3, 0)\}$, (3) $\{(3, -2)\}$, (4) $\{(-3, 2)\}$,
(5) $\{(3.5, 2.5)\}$, (6) $\{(0, 3)\}$.

Exercise 4.5

- (1) 35, 20. (2) 26, 63 year's. (3) 58. (4) 40 paise, 55 paise. (5) 15 cm., 8 cm.
(6) 90 km p. h. and 75 km p.h. (7) 12. (8) 176. (9) Rs. 4.80. (10) 52, 38.
(11) 300 of Rs. 2/- and 700 of Rs. 1/- (12) At a distance of 0.9 decemeter from
the end having a weight of 1.4 Kg. (13) 16 of 10 paise and 14 of 20 paise.
(14) 140 of 10 paise and 80 of 25 paise. (15) 680 km. (16) 320, 400. (17) 240, 320
(18) Rs. 40. (19) 448 km. (20) 94, 86.

Exercise 5.1

- (1) 13. (2) 7^{10} . (3) base, index (4) 3. (5) -7 . (6) 1. (7) x^4 . (8) -9 . (9) 9.
 (10) -5 . (11) -28 . (12) 15. (13) -21 . (14) $-3, -3$. (15) 7, 7. (16) $-6, 9$.
 (17) 8, 24, $-8, -12$.

Exercise 5.2

- (1) (i) $\sqrt[4]{12^3}$, (ii) $\sqrt[4]{23}$, (iii) $\sqrt[3]{\frac{1}{7^3}}$, (iv) $\sqrt[3]{9}$, (v) $\sqrt[4]{29^2}$, (vi) $\sqrt{9^3}$,
 (vii) $\frac{1}{\sqrt[3]{xm}}$, (viii) $\sqrt{m^p}$, (ix) $\sqrt{\left(\frac{m}{n}\right)^7}$, (x) $\frac{1}{\sqrt[3]{x^4}}$.
 (xi) $\sqrt[4]{31}$, (xii) $\sqrt[4]{32}$, (xiii) $\sqrt[3]{12}$.
 (2) (i) $2^{3/4}$, (ii) $8^{3/7}$, (iii) $\left(\frac{7}{4}\right)^{2/5}$, (iv) $x^{-3/2}$, (v) $x^{-7/5}$, (vi) $13^{1/3}$.
 (vii) $27^{1/2}$, (viii) $\left(\frac{4}{3}\right)^{4/6}$, (ix) $x^{p/q}$, (x) $\left(\frac{x}{y}\right)^{9/8}$, (xi) $137^{1/7}$,
 (xii) $17^{1/2}$, (xiii) $24^{1/3}$, (xiv) $30^{1/4}$.

Exercise 5.3

- (1) (i) $\sqrt{25^3}, 125$, (ii) $\sqrt[4]{16}, 2$, (iii) $\sqrt[3]{32^3}, 8$, (iv) $\sqrt{\left(\frac{49}{25}\right)^3}, \frac{343}{125}$,
 (v) $\sqrt{4^3} = 8$, (vi) $\frac{1}{\sqrt{9^3}}, \frac{1}{27}$, (vii) $\sqrt[3]{\frac{1}{8^2}}, \frac{1}{4}$,
 (viii) $\sqrt{\left(\frac{4}{9}\right)^3}, \frac{8}{27}$, (ix) $\frac{1}{\sqrt{\left(\frac{4}{9}\right)^3}}, \frac{27}{8}$, (x) $\sqrt[4]{64}, 2$,
 (xi) $\sqrt[4]{625}, 5$, (xii) $\sqrt[3]{27} \times \sqrt{4}, 6$, (xiii) $\sqrt[3]{8^2} \times \sqrt{4^3}, 32$,
 (xiv) $\frac{\sqrt[3]{125^2}}{\sqrt[4]{16^3}}, \frac{25}{8}$, (xv) $\frac{\sqrt{49} \times \sqrt[3]{8^3}}{\sqrt{4^3}}, \frac{7}{2}$, (xvi) $(\sqrt{9^3})^2, 729$,
 (xvii) $\sqrt{4} \times \sqrt[3]{8}, 4$, (xviii) $\frac{1}{\sqrt{25^3}}, \frac{1}{125}$, (xix) $\sqrt[4]{16} \times \sqrt{16}, 8$,
 (xx) $\frac{\sqrt[3]{81^3}}{\sqrt[4]{81}}, 9$, (xxi) $(\sqrt{4} \times \sqrt[3]{125})^2, 100$, (xxii) $\left(\frac{\sqrt{9}}{\sqrt{4}}\right)^2, \frac{9}{4}$.
 (2) (i) 27, (ii) 64, (iii) 27, (iv) 128, (v) 729.

Exercise 5.4

- (1) (i) $\frac{1}{8^{2/3}}, \frac{1}{4}$, (ii) $125^{1/3}, 5\sqrt{5}$, (iii) $\frac{1}{9^{2/3}}, \frac{1}{3\sqrt[3]{3}}$,
 (iv) $\frac{1}{4^{3/2}}, \frac{1}{8}$, (v) $(8)^{2/3}, 4$, (vi) $\frac{1}{(81)^{3/4}}, \frac{1}{27}$,
 (vii) $\frac{1}{4^{2/7}}, \frac{1}{\sqrt[4]{16}}$, (viii) $\frac{1}{5^{3/4}}, \frac{1}{\sqrt[4]{125}}$, (ix) $\frac{1}{10^{3/5}}, \frac{1}{\sqrt[5]{1000}}$,
 (x) $\frac{1}{9^{3/2}}, \frac{1}{27}$, (xi) $9^{3/2}, 27$, (xii) $x^{7/5}, \sqrt[5]{x^7}$, (xiii) $(xy)^{4/7}, \sqrt[7]{(xy)^4}$,
 (xiv) $49^{3/2}, 343$, (xv) $a^{7/8}, \sqrt[8]{a^7}$, (xvi) $x^{7/20}, \sqrt[20]{x^7}$,
 (xvii) $\frac{1}{a^{4/3}}, \frac{1}{b^{5/4}}, \frac{1}{\sqrt[2]{a^2} \times b^5}$, (xviii) x^3b^5 , (xix) $\frac{1}{x^{2/3}} + \frac{1}{y^{2/3}} = \frac{1}{a^{2/3}}$,
 (xx) $\frac{1}{(x+y)(y+z)^2}$.

- (2) (i) $\frac{1}{13^{-2/3}}$, (ii) $5^{-2/7}$, (iii) $a^{-3/4} b^{-2/3}$, (iv) $\frac{1}{x^{-4/7}}$,
 (v) $\frac{1}{x^{-3/2} y^{-4/7}}$, (vi) $\frac{3^{-y}}{2^{-x}}$, (vii) $\frac{1}{\left(\frac{2}{3}\right)^{-2/7}}$, (viii) $\left(\frac{5}{7}\right)^{-3/5}$,

- (ix) $\frac{1}{\left(\frac{1}{3}\right)^{-4/3}}$, (x) $x^{-4/5}$.

- (3) (i) 2^{20} , (ii) 3^{12} , (iii) 2^{11} , (iv) $\frac{1}{5^{12}}$, (v) $\frac{1}{2^8}$,
 (vi) $\frac{1}{2^{12}}$, (vii) 27 , (viii) $\frac{1}{5^7}$, (ix) x^3 ,
 (x) $\frac{1}{a^3}$, (xi) 2^{14} , (xii) 3^9 , (xiii) b^6 , (xiv) $\frac{1}{x^{9/2} y^9}$,
 (xv) $m^5 n^{12}$, (xvi) $\frac{a^2}{b^6}$, (xvii) $\frac{1}{x^5 y^{12}}$, (xviii) $\frac{a}{b^2}$, (xix) $\frac{1}{x^4 y^2}$, (xx) $\frac{2^4}{3^6}$.

- (4) (i) $\sqrt{2}$, (ii) $\sqrt{5}$, (iii) $\sqrt[12]{x^7}$, (iv) b^2 ,
 (v) $\frac{1}{\sqrt[3]{a^5}}$, (vi) $\frac{1}{\sqrt[4]{x^{11}}}$, (vii) $\frac{1}{\sqrt{x^7}}$.

Exercise 5.5

- (1) (i) $2^{7/8}$, (ii) $\frac{1}{5^{7/8}}$, (iii) $\frac{x^{17/16}}{y^{13/6}}$, (iv) a^2 , (v) $\frac{1}{x^{1/12}}$,
 (vi) $\frac{x^{7/3}}{y^{7/16} z^{89/30}}$, (vii) $\frac{1}{x^{5/2}} + \frac{1}{x^2} + \frac{1}{x^{3/2}} + \frac{1}{x}$.
- (2) (i) $x + x^{1/2} + x^3$, (ii) $a^{1/6} - \frac{2}{a^2}$, (iii) $\frac{1}{a^{1/6} b^{1/6}} - a^{5/6} b^{5/6} + \frac{1}{a^{1/3} b}$,
 (iv) $1 - x^{1/2} y^{2/3} + \frac{y^{1/3}}{x^{1/2}} - y$, (v) $2a^{2/3} + 6ab - \frac{b^{1/3}}{a^{1/3}} - 3b^{4/3}$,
 (vi) $x^2 + 2xy^{1/2} + y$, (vii) $x - 3x^{2/3} y^{1/3} + 3x^{1/3} y^{2/3} - y$,
 (viii) $x + 2 + \frac{1}{x}$.

Exercise 5.6

- (1) (i) $5^{26/3}$, (ii) $\frac{1}{a^{4/76}}$, (iii) $a^{4/9}$, (iv) $\frac{1}{x^{4/3}}$,
 (v) $a + b$, (vi) 27, (vii) 125, (viii) $x^{2/3} - y^{2/3}$,
 (ix) 16, (x) 81, (xi) $5^{22/3}$, (xii) $2^5 \times 3^{16/3}$,
 (xiii) 1, (xiv) $\frac{x^{67/30}}{y^5 z^{13/3}}$, (xv) $\frac{25}{8}$,
 (xvi) $\frac{7}{2}$, (xvii) 1.
- (2) (i) $x^{2/5} + 2x^{1/5}y^{1/5} + y^{2/5}$, (ii) $x^{4/3} - 2 + \frac{1}{x^{4/3}}$,
 (iii) $x^4 + 3x^{8/3}y^{4/3} + 3x^{4/3}y^{8/3} + y^4$,
 (iv) $x + 3x^{1/3} + \frac{3}{x^{1/3}} + \frac{1}{x}$,
 (v) $\frac{x}{y} - \frac{3x^{1/3}}{y^{1/3}} + \frac{3y^{1/3}}{x^{1/3}} - \frac{y}{x}$.
- (3) (i) $(x^{1/2} + y^{1/3})^2$, (ii) $(x^{1/6} + y^{1/6})^3$, (iii) $(a^{1/4} - a^{1/4})^2$,
 (iv) $(x^{1/3} - x^{-1/3})^3$, (v) $(a^{1/2} + a^{-1/2})^2$.
- (4) (i) $(x^{1/2} + y^{1/2})(x^{1/2} - y^{1/2})$, (ii) $(x^{1/6} + y^{1/6})(x^{1/6} - y^{1/6})$,
 (iii) $(x^{3/10} + y^{3/10})(x^{3/10} - y^{3/10})$, (iv) $(x^{-1/2} + y^{-1/2})(x^{-1/2} - y^{-1/2})$.
- (5) (i) $(x^{1/3} - y^{1/3})(x^{2/3} + x^{1/3}y^{1/3} + y^{2/3})$,
 (ii) $(x^{1/6} + y^{1/6})(x^{1/3} - x^{1/6}y^{1/6} + y^{1/3})$,
 (iii) $(x^{1/9} + y^{1/9})(x^{2/9} - x^{1/9}y^{1/9} + y^{2/9})$,
 (iv) $(x^{1/6} - y^{1/6})(x^{1/3} + x^{1/6}y^{1/6} + y^{2/9})$.
- (6) (i) $3^{10/30}, 2^{15/30}, 5^{6/30}$, (ii) $3^{20/60}, 5^{45/60}, 7^{24/60}$, (iii) $x^{6/12}, x^{9/12}, x^{4/12}$.

Exercise 5.7

$$\begin{array}{lll}
 (1) \frac{5^{2/7}}{2^{9/7} \times 7^{12/35}} & (2) \frac{5}{2^{2/9} \times 7^{-4/3}} & (3) \frac{x^{9/2} z^{3/2}}{8y} \\
 (4) \frac{b^{1/2} c d^{1/6}}{a^{1/6}} & (5) \frac{b^{1/18} c^{1/8}}{a^{17/18}} & (6) \frac{b^{14/9} c^{11/6}}{o^{3/6}} \\
 (7) \frac{1}{x^{3/4} y^{8/5} z^{9/50}} & (8) \frac{4x^{9/2}}{9y^3 z^2} &
 \end{array}$$

Exercise 5.8

$$\begin{array}{lll}
 (1) \text{ (i) } 2^{2/3}, 7^{1/3}, 5^{1/2} & \text{(ii) } 2^{1/3}, 3^{1/4}, 5^{1/5} & \text{(iii) } 7^{-2/3}, 6^{-1/2}, 5^{-1/3} \\
 & \text{(iv) } 4^{-1/3}, 3^{-1/3}, 2^{-1/2} & \\
 (2) \text{ (i) } \{-2\} & \text{(ii) } \{\frac{11}{5}\} & \text{(iii) } \left\{ \frac{-17}{10} \right\}, \text{ (iv) } \{5\}, \\
 & \text{(v) } \left\{ \frac{-27}{11}, \frac{12}{11} \right\} & \text{(vi) } \left\{ \frac{11}{3}, -4 \right\}, \text{ (vii) } \left\{ \frac{-7}{3}, \frac{1}{4}, 0, 14 \right\}, \\
 & \text{(viii) } \{-1, \frac{1}{2}, 2\} & \text{(ix) } \{0, 1\}, \text{ (x) } \{1, 9\}, \text{ (xi) } \{\frac{1}{8}, 27\}, \text{ (xii) } \{\frac{2}{3}\},
 \end{array}$$

Exercise 5.9

$$\begin{array}{lll}
 (1) \text{ (i) } 3^{36} & \text{(ii) } \frac{1}{50} & \text{(iii) } \frac{1}{x^2}, \text{ (iv) } 2^{5/6}, \text{ (v) } \frac{1}{2}, \text{ (vi) } 14, \\
 & \text{(vii) } xyz, & \text{(viii) } x, \text{ (ix) } \frac{1}{9} \\
 (2) \text{ (i) } \{3\} & \text{(ii) } \left\{ -\frac{1}{11} \right\} & \text{(iii) } \left\{ \frac{3}{2} \right\}, \text{ (iv) } \left\{ -2, \frac{-1}{2}, 1 \right\}, \\
 & \text{(v) } \{8, 27\} & \text{(vi) } \left\{ 1, \frac{1}{4} \right\} \\
 (3) \text{ (i) } x^{4/3} + 2x + x^{2/3} & \text{(ii) } x - 2 + \frac{1}{x} & \\
 & \text{(iii) } x^2 - 3x^{2/3} + \frac{3}{x^{2/3}} - \frac{1}{x^2} & \text{(iv) } \frac{x}{y} + 2 + \frac{y}{x}, \\
 & \text{(v) } \frac{4}{3} + 3\sqrt[3]{3} + \frac{3}{\sqrt[3]{3}} & \\
 (4) \text{ (i) } (x^{1/2} + x^{-1/2})^2 & \text{(ii) } (x^{2/3} + x^{-2/3})^3 & \\
 & \text{(iii) } \left(\frac{a^{1/2}}{b^{1/2}} - \frac{b^{1/2}}{a^{1/2}} \right)^2 & \text{(iv) } \left(\frac{x^{1/3}}{y^{1/3}} - \frac{y^{1/3}}{x^{1/3}} \right)^2
 \end{array}$$

- (5) (i) $(x^{-1/2} + y^{-1/2}) (x^{-1/4} + y^{-1/4}) (x^{-1/4} - y^{-1/4})$,
 (ii) $(x^{1/8} + y^{1/8}) (x^{1/16} + y^{1/16}) (x^{1/16} - y^{1/16})$,
 (iii) $(x^{3/10} + y^{3/10}) (x^{3/20} + y^{3/20}) (x^{3/20} - y^{3/20})$.
- (6) (i) $(x^{-1/3} - y^{-1/6}) (x^{-2/3} + x^{-1/3} y^{-1/6} + y^{-1/3})$,
 (ii) $(x^{1/6} - y^{-1/6}) (x^{1/3} + x^{1/6} y^{-1/6} + y^{-1/3})$,
 (iii) $(x^{1/9} + y^{-1/3}) (x^{2/9} - x^{1/9} y^{-1/3} + y^{-2/3})$,
 (iv) $(x^{2/9} + y^{1/2}) (x^{4/9} - x^{2/9} y^{1/2} + y)$.
- (7) (i) $\sqrt[3]{4}$, $\sqrt[4]{6}$, $\sqrt{2}$. (ii) $\sqrt[12]{125}$, $\sqrt[8]{10}$, $\sqrt[3]{3}$. (iii) $5^{1/2}$, $10^{1/4}$, $4^{1/3}$.

Exercise 6.1

- (1) (i) $\log_{10} 100 = 2$. (ii) $\log_5 125 = 3$. (iii) $\log_2 16 = x$.
 (iv) $\log_a 27 = 3$, (v) $\log_{10} m = y$, (vi) $\log_5 1 = 0$,
 (vii) $\log_7 7 = 1$, (viii) $\log_b a = x$, (ix) $\log_n m = x + y$,
 (x) $\log_{10} 37.81 = 1.5776$, (xi) $\log_{10} 0.001 = -3$,
 (xii) $\log_{10} 20 = 1.3010$.
- (2) (i) $5^4 = 625$, (ii) $a^k = 5$, (iii) $5^1 = 5$, (iv) $7^0 = 1$,
 (v) $10^{-2} = 0.01$, (vi) $k^c = x + y$, (vii) $b^p = m$,
 (viii) $10^{-1} = 0.1$, (ix) $10^3 = 1000$, (x) $10^{0.3762} = 2.378$,
 (xi) $10^{1.5776} = 37.81$.
- (3) (i) 2, (ii) 4, (iii) 2, (iv) 2, (v) 6, (vi) 3, (vii) 2,
 (viii) 1, (ix) -1, (x) -2, (xi) 2, (xii) 4, (xiii) 3,
 (xiv) 0, (xv) 1.
- (4) (i) 3, (ii) 1, (iii) -29, (iv) 4, (v) 6, (vi) 0, (vii) 0,
 (viii) 24, (ix) 2, (x) $\frac{2}{3}$.
- (5) Right (i), (iv), (v), (viii), (ix), (x), (xiii), (xiv),
 (xvi), (xviii), (xx), (xxii), (xxiii).
 Wrong (ii), (iii), (vi), (vii), (xi), (xii), (xv), (xvii), (xix), (xxi),
 (xxiv), (xxv), $1 \leftrightarrow e$, $2 \leftrightarrow F$, $3 \leftrightarrow A$, $4 \leftrightarrow E$, $5 \leftrightarrow F$.

Exercise 6.2

- (1) (i) 2, 3. (ii) 2, -4. (iii) 2, 5. (iv) 8, (v) {4},
 (vi) $\left\{ -\frac{18}{5} \right\}$.
- (2) (i) $a = 4b$, (ii) $a^2 + b^2 = 0$.

Exercise 6.3

- (1) (i) 2, (ii) $\log_5 175$, (iii) $\log_5 \frac{8}{5}$, (iv) $\log_a 224$, (v) 0,
 (vi) $\log_n xyz$, (vii) $\log_x ab$, (viii) 0.
 (2) (i) $\log_2 3 + \log_2 5$, (ii) $\log_4 3 + \log_4 7$, (iii) $\log_5 2 + \log_5 3 + \log_5 11$,
 (iv) $\log_5 7 + \log_5 7$, (v) $\log_7 5 + \log_7 5 + \log_7 5$,
 (vi) $\log_2 2 + \log_2 2 + \log_2 2 + \log_2 2 + \log_2 2$, (vii) $\log_5 a + \log_5 b$,
 (viii) $\log_x (a + b) + \log_x (c + d)$,
 (ix) $\log_m (a + b) + \log_m (a - b)$,
 (x) $\log_m x + \log_m (y^{-1})$, (xi) $\log_x (a + b) + \log_x (c + d)^{-1}$,
 (xii) $\log_n 3.78 + \log_n 5.13$,
 (xiii) $\log_a 7.132 + \log_a 4.7 + \log_a 81.23$,
 (xiv) $\log 3.4 + \log 3.4 + \log 3.4$.
 (3) (i) 74.37, (ii) 8.026, (iii) 51.78, (iv) 18.61, (v) 166.3.

Exercise 6.4

- (1) (i) 1, (ii) $\log_5 \frac{17}{7}$, (iii) $\log_m \frac{a}{b}$, (iv) $\log_a \frac{x}{y^2}$,
 (v) $\log_m (a - b)$, (vi) -5, (vii) $-\log_a y^2$,
 (viii) $\log_k (a^2 - b^2)$, (ix) $\log_k \frac{a^2}{b^2}$, (x) $\log_k x^2$.
 (2) (i) $\log_a 13 - \log_a 7$, (ii) $\log_a x - \log_a y$,
 (iii) $\log_{10} 27.32 - \log_{10} 8.437$, (iv) $\log_{10} 2 + \log_{10} 7 - \log_{10} 5$,
 (v) $\log_a x + \log_a y - \log_a z$,
 (vi) $\log_{10} 2.314 + \log_{10} 37.92 - \log_{10} 48.33$,
 (vii) $\log_{10} 3 - \log_{10} 8 - \log_{10} 5$, (viii) $\log_k a - \log_k b - \log_k c$,
 (ix) $\log_{10} 99.07 - \log_{10} 0.03 - \log_{10} 0.532$,
 (x) $\log_{10} 3 + \log_{10} 5 - \log_{10} 7 - \log_{10} 8$,
 (xi) $\log_{10} 8.33 + \log_{10} 7.5 - \log_{10} 2.8 - \log_{10} 0.38$,
 (xii) $\log_{10} 0.03 + \log_{10} 7.089 - \log_{10} 2.7 - \log_{10} 0.532$.
 (3) (i) 7.863, (ii) 6.747, (iii) 1.004, (iv) 4.677.

Exercise 6.5

- (1) (i) $\log_a \left(\frac{x^3 z^5}{y^2} \right)$, (ii) $\log_a 2$, (iii) $\log_a (125)$, (iv) $\log_a (81)$,
 (v) $\log_5 (a^3 + b^3)$.

$$(3) \text{ (i) } 3 \log_a \left(\frac{x}{y} \right) \quad \text{(ii) } \frac{1}{3} \log_m a + \frac{2}{3} \log_m b.$$

$$\text{(iii) } \frac{1}{5} \log_k x + \frac{2}{5} \log_k y, - \frac{3}{5} \log_k z.$$

$$\text{(iv) } 2 \log_{10} (3.27) + \frac{1}{3} \log_{10} (0.38)^6.$$

$$\text{(v) } 3 \log_{10} (6.45) + \frac{1}{3} \log_{10} (0.08) - \frac{2}{3} \log_{10} (7.25) - \log_{10} (4.23).$$

$$(4) \text{ (i) } 22.76, \quad \text{(ii) } 2.797, \quad \text{(iii) } 1.119.$$

Exercise 6.6

- (1) (i) 1, (ii) $\overline{2}$, (iii) 0, (iv) $\overline{3}$, (v) $\overline{1}$, (vi) 2, (vii) 0, (viii) $\overline{4}$,
 (ix) 2, (x) 3, (xi) 0, (xii) $\overline{1}$, (xiii) 1, (xiv) 0, (xv) $\overline{3}$, (xvi) 2,
 (xvii) 2, (xviii) $\overline{2}$, (xix) 1, (xx) 1, (xxi) 0.

(2) **Characteristic : 1**

27.01, 23.00, 10.10, 27.895.

Characteristic : $\overline{2}$

0.02701, 0.023, 0.01010, 0.080, 0.027895.

Characteristic : 0

2.701, 2.3, 1.01, 8.0, 2.7895.

Characteristic : 2

270.1, 230, 101.0, 800, 278.95.

Characteristic : $\overline{1}$

0.2701, 0.23, 0.1010, 0.80, 0.27895.

Characteristic : 4

27010, 23000, 10100, 80000, 27895.

Characteristic : $\overline{3}$

0.002701, 0.0023, 0.001010, 0.0080, 0.0027895.

Characteristic : 3

2701, 2300, 1010, 8000, 2789.5.

Characteristic : $\overline{4}$

0.0002701, 0.00023, 0.0001010, 0.0008, 0.00027895.

Exercise 6.7

- | | | | |
|----------------------------|--------------|----------------------------|----------------------------|
| (1) 0.3858, | (2) 1.1810, | (3) 0.9115, | (4) $\overline{3}.5845$, |
| (5) 1.0792. | (6) 1.5378, | (7) 0.6990. | (8) $\overline{2}.9280$. |
| (9) $\overline{3}.4364$. | (10) 5.6414. | (11) 1.9956. | (12) $\overline{2}.9591$. |
| (13) 1.6305. | (14) 0.6305. | (15) $\overline{1}.6305$. | (16) $\overline{2}.6305$, |
| (17) $\overline{3}.6305$. | (18) 2.6305. | (19) 1.9634. | (20) $\overline{2}.9634$. |
| (21) $\overline{1}.3779$. | (22) 0.3779. | (23) 5.7354. | (24) 6.9412. |
| (25) 0.9031. | (26) 1.9191. | (27) 0.9877. | (28) 1.4772. |
| (29) 3.3010. | (30) 1.3010. | | |

Exercise 6.8

- | | | | |
|-------------|----------------|--------------|-------------|
| (1) 87.22. | (2) 1.006. | (3) 0.02736. | (4) 5.156. |
| (5) 0.1882. | (6) 216.2. | (7) 2.162. | (8) 0.2162. |
| (9) 21.62. | (10) 0.02162. | (11) 1714. | (12) 74.67. |
| (13) 1.001. | (14) 0.001210. | (15) 1. | (16) 0. |
| (17) 10.23. | (18) 303.9. | | |

Exercise 6.9

- | | | | |
|-------------|--------------|----------------|-----------------|
| (1) 981.1. | (2) 30.73. | (3) 34.02. | (4) 2.054. |
| (5) 220.5. | (6) 2.314. | (7) 0.1383. | (8) 0.00009488. |
| (9) 0.4357. | (10) 0.9122. | (11) 0.009059. | (12) 0.05097. |
| (13) 1.259. | (14) 206.5. | (15) 0.1232. | |

Exercise 7.1

- (1) (i) 3, 4. (ii) 5, -7. (iii) $12, \frac{9}{13}$. (iv) 1, -72. (v) 0, 100.
 (vi) n, m. (vii) 1000, 2767. (viii) No, 0.

(2) $6x^5, -4x^3, \frac{7}{12}x, \sqrt[4]{10}$.

(3) $7x^3, \frac{6}{11}x^5, -5x, 4x, -10$.

- (4) (i) 2, -5, (ii) whole (iii) 0, (iv) zero, (v) 1,
 (vi) -1, (vii) 0, (viii) $-5x^{-2}$ because -2 the index of x is not a
 whole number, (ix) $\sqrt[5]{x}$ because $\frac{1}{5}$ the index of x is not the whole number.
 (x) $\frac{3}{x}$ because -1, the index of x , is not a whole number.

Exercise 7.2

- (1) (i) 2, (3, -4, 5). (ii) 3, (6, -5, 7, 3), (iii) 4, (7, 0, -3, 2, 0).
 (iv) 7, (-1, 0, 0, 5, 0, -5, 0, 0). (v) 1, (3, 4). (vi) 1, (-5, 9).
 (vii) 1, (7, 0). (viii) 0, (13). (ix) 4, (1, 0, 0, 0, 0).
 (x) 3, (1, 0, 0, -1). (xi) 0, (-5). (xii) 4, (-3, 0, 0, 0, 4).
 (xiii) 0, (1), (xiv) 1, (1, 0). (xv) 4, (a, -1, 0, 2, -b).
 (xvi) 7, (-1, p, 0, -7, 0, 0, q, -1).
- (2) (i) $2, 3x^2 + 5x + 1$ (ii) $2, 7x^2 - 2x + 4$. (iii) $1, x + 1$. (iv) $1, x$.
 (v) $2, x^2$, (vi) $3, -3x^3 - 3$. (vii) $0, 5$,
 (viii) $6 - \frac{5}{2}x^6 + \frac{1}{2}x^5 + \frac{4}{7}x^3 - \frac{5}{8}$. (ix) $3, 3x^3$.
 (x) $4, ax^4 + bx^3 + cx^2 + dx + e$, (xi) $4, ax^4 + x^3 - 2x^2 + b$.
 (xii) $6, x^6 + px^5 - 4x^3 + qx - 1$.
- (3) $a = 5, b = 0, c = 3, d = -7, e = 8$.

Exercise 7.3

- (1) $8x^2 + 15$. (2) $x^2 + 9x$, (3) $2x^3 - x$. (4) $x^4 + 2x^2 + 2x + 3$.
 (5) x^5 , (6) $-2x^3 - 8x^2 + 7x - 1$, (7) $7x^2 + 5x + 8$. (8) $5x^2 - 3x - 11$.
 (9) $3x^2 + 2x - 7$. (10) 0. (11) 0. (12) 0,
 (13) $ax^3 + bx^2 + cx + d$. (14) 0.
 (15) $(a + 5)x^4 + (b - 3 + q)x^2 + (c + p - 7)$.
 (16) $-3x^4 - 4x^3 + 4x^2 + x + a$.

Exercise 7.4

- (1) (i) $3x^2 - 5x$, (ii) $6x^4 + 6x^3 - 7x^2 + 3x - 8$, (iii) $4x^3 + x^2 + 8x - 11$,
 (iv) $-10x^2 + 3x + 7$, (v) $-6x^3 - 4x^2 + 2x - 8$,
 (vi) $5x^6 - 3x^5 - 2x^4 - 3x^3 + x^2 + x - 7$,
 (vii) $6x^4 - 3x^3 + 9x^2 - (p + q)x + (q - p)$,
 (viii) $(a - p)x^2 + (b - q)x + (c - r)$.
- (2) (i) $6x^2 - 6x$, (ii) $3x^2 - 10x + 3$, (iii) $3x^4 - 3x^3 - 5x^2 + 15x + \frac{1}{4}$,
 (iv) $-2x^4 - 65x^3 + 80x^2 + 25x - 27$, (v) $-2x^3 + 6x^2 + 5x - 30$.
- (3) 5. (4) $-x^4 + 2x^3 + 2x - 2$. (5) $6x^3 + 4x^2 - 11x + 7$.
 (6) $-x^4 - 5x^3 - 5x^2 - 7x + 11$. (7) $5x^3 - x^2 + 9x - 1$. (8) $-8x^2 + 2x - 3$.
 (9) $2x^3 + 4x^2 - 2x - 17$. (10) 0.
 (11) $a = 7, b = 4, c = -6, d = 16, e = -19$. (12) $x^2 + 5x - 6$.
 (13) $6a + 2$. (14) $2m^3 - 7m$. (15) $\frac{1}{3}a^2 + \frac{1}{2}a + 8$. (16) (i) 0. (ii) $-c$.
 (17) If c is the polynomial of degree Zero then its opposit is $-c$.
 (18) The number of terms of polynomial is more than its degree by 1.
 (19) Equality relation. (20) These polynomials are opposites of each other.

Exercise 7.5

- (1) $3x^3 + 14x^2 + 13x + 20, 3.$ (2) $6x^4 - 12x^3 + x^2 + 2x - 8, 4.$
 (3) $-6x^5 + 13x^3 - 8x^2 - 6x + 12, 5.$ (4) $4x^4 + 14x^3 + 13x^2 + 11x + 3, 4.$
 (5) $-35x^4 + 27x^3 + 42x^2 - 6x - 4, 4.$ (6) $x^3 - 8, 3.$
 (7) $x^4 - a^4, 4.$ (8) $x^5 + p^5, 5.$ (9) $8x^3 - 27, 3.$
 (10) $4ax^{10} + 24x^8 - 3ax^7 + (6p + 4b)x^6 - 34x^5 + bpx^4 + (-3b + 20)x^3 - 12x^2 + 5px - 15, 10.$
 (11) $apx^{10} + bpx^9 + (aq + pd)x^8 + (bq - 7a)x^5 + (ar - 7b)x^4 + brx^3 + dqx^2 - 7dx + dr, 10.$
 (12) $7mx^5 + (7n - 5m)x^4 + (7p - 5n + am + m)x^3 + (-5p + an + n)x^2 + (ap + p)x, 5.$
 (13) $125x^6 + 200x^4 + 60x^2 + 64, 6.$
 (14) $12x^5 - 58x^4 - 50x^3 + 103x^2 - 64x + 21, 5.$
 (15) $3x^6 - 17x^5 + 31x^4 - 10x^3 - 42x^2 + 31x - 4, 6.$
 (16) $5ax^9 - 3ax^8 - 20x^7 + (5p + 12)x^6 - 3px^5 + 8ax^4 - 9ax^3 - 32x^2 + (8p + 36)x - 9p, 9.$
 (17) $m + n.$ (18) Multiplicative identity.
 (19) The product of any polynomial and the zero polynomial is the zero polynomial.

Exercise 7.6

- (1) Quotient $4x + 2$; Remainder 0. (2) Quotient $2x^3 + x^2 - 4x$; Remainder 0.
 (3) Quotient $x + 2$; Remainder 0. (4) Quotient $x + 4$; Remainder 0.
 (5) Quotient $5x$; Remainder 0. (6) Quotient $12x - \frac{3}{2}$; Remainder 3.
 (7) Quotient $x^2 - x + 1$; Remainder 0.
 (8) Quotient $3x^2 + 10x + 4$; Remainder + 10.
 (9) Quotient $5x - 4$; Remainder 32. (10) Quotient $x^2 - x - 1$; Remainder - 2.
 (11) Quotient $\frac{4}{3}y + \frac{4}{3}$; Remainder $\frac{1}{3}y + \frac{1}{3}.$
 (12) Quotient $2x - 3$; Remainder 0.
 (13) Quotient $2y^2 - \frac{8}{3}y + \frac{56}{9}$; Remainder $-\frac{299}{9}y^2 - \frac{7}{9}.$
 (14) „ $3x^2 + \frac{3}{4}x - \frac{41}{16}$; Remainder $\frac{103}{16}.$
 (15) „ $3x^3 - 6x^2 + 12x - 27$; Remainder 61.
 (16) „ $3x^3 + 4x + 8$; Remainder $10x^2 - 13x - 19.$
 (17) „ $x^2 + 2x + 4$; Remainder 0.

- (18) Quotient $x^3 + ax^2 + a^2x + a^3$; Remainder 0.
 (19) „ $x^2 + 2ax + a^2$; Remainder 0.
 (20) „ $ax + b$; Remainder $ac - b^2$.
 (21) „ $ax + (2b - 3a)$; Remainder $9a - 6b + c$.
 (22) „ $x^5 + 2ax^4 + 4a^2x^3 + 8a^3x^2 + 16a^4x + 32a^5$; Remainder 0.
 (23) „ $3x - 5$; Remainder 0.
 (24) „ $ax + (ap - b)$; Remainder $ap^2 + bp + c$.

Exercise 7.7

- (1) Quotient $x^2 - 4x$; Remainder 7.
 (2) „ $3x^2 + 16x + 41$; Remainder 84.
 (3) „ $5x^2 - 22x + 44$; Remainder - 104.
 (4) „ $2x^2 - 6x + 14$; Remainder - 39.
 (5) „ $3x^3 + 4x^2 + 11x + 51$; Remainder 203.
 (6) „ $3x^3 - 4x^2 + 25x - 93$; Remainder 370.
 (7) „ $x^3 - 3x^2 + 2x - 4$; Remainder 2.
 (8) „ $x^4 - 5x^3 + 8x^2 - 3x + 4$; Remainder - 13.
 (9) „ $x^3 + 3x^2 + 9x + 27$; Remainder 0.
 (10) „ $x^3 - 4x^2 + 16x - 64$; Remainder 320.
 (11) „ $x^4 + 2x^3 + 4x^2 + 8x + 16$; Remainder 0.
 (12) „ $3x^2 + 10x + 4$; Remainder 10.
 (13) „ $x^2 - 3x + 3$; Remainder - 4.
 (14) „ $3x^3 - 6x^2 + 12x - 27$; Remainder 61.
 (15) „ $3x - 5$; Remainder 0.
 (16) „ $x^2 + 2ax + a^2$; Remainder 0.
 (17) „ $ax + (2b - 3a)$; Remainder $9a - 6b + c$.
 (18) „ $ax + (ap - b)$; Remainder $ap^2 + bp + c$.
 (19) „ $3x^3 + 4x^2 + 2x + 23$; Remainder 36.
 (20) „ $15x^3 + 39x^2 + 112x + 336$; Remainder 996.
 (21) „ $7x^3 + 30x^2 + 150x + 805$; Remainder 4040.
 (22) „ $4x^4 + 8x^3 + 13x^2 + 27x + 54$; Remainder 103.
 (23) „ $25x - 100$; Remainder 416.
 (24) „ $12x^3 + x + 1$; Remainder 9.

Exercise 7.8

Divisible :—(1), (2), (3), (7), (11), (12), (13), (14), (15).
 Not divisible :—(4), (5), (6), (8), (9), (10).

Exercise 7.9

- (1) (i) 8, (ii) -4 , (iii) -2 , (iv) $a^2 + 3a - 2$,
 (v) $4b^2 + 6b - 2$, (vi) $(a+b)^2 + 3(a+b) - 2$.
- (2) (i) $\frac{1}{2}$, (ii) $\frac{3}{2}$, (iii) 3, (iv) $-\frac{27}{2}$, (v) $-2a^2 - \frac{1}{2}a + 3$,
 (vi) $-\frac{b^3}{2} - \frac{b}{2} + 3$.
- (3) (i) -495 , (ii) 0, (iii) 3, (iv) $-4c^3 + c$, (v) $-108c^3 + 3c$,
 (vi) $-4a^3 - 11a - \frac{11}{a} - \frac{4}{a^3}$.
- (4) (i) 4, (ii) -58 , (iii) 5, (iv) $3d^3 + 2d^2 + 5$,
 (v) $3a^3 + 2a^2 - 9a + 1 + \frac{9}{a} + \frac{2}{a^2} - \frac{3}{a^3}$.
- (5) (i) $-5a^3$, (ii) $13a^3$.
 (6) (i) $a^2 + 4a + 4$, (ii) $a^2 + 2a + 1$, (iii) 1.
 (7) (i) $x^5 - 7$, (ii) $3x^5 - 2x^2 + 4x - 7$. (8) (i) $x + 1$, (ii) $2x - 1$.
- (9) $a = 2$, (10) $b = 5$, (11) $a = -\frac{2}{7}$, $b = \frac{74}{7}$, (12) $p = -7$, $q = 24$.

Exercise 7.10

- (1) -4 , (2) 69, (3) 0, (4) -4 , (5) 0.

Exercise 7.11

- (1) 23, (2) 41, (3) 47, (4) -36 , (5) 2, (6) -75 , (7) 0.
 (8) $2a^2 - 4a + 3$, (9) 0, (10) -4 , (11) 14, (12) 356, (13) 0.
 (14) 0, (15) 0, (16) 0, (17) 6, (18) 92, (19) 50, (20) -16 .
 (21) 68, (22) $p = -10$, (23) $p = 11$, (24) $p = -5$, $q = 1$.
 (25) $a = 0$, $b = 9$, (26) $a = 7$.

Exercise 7.12

- (1) Rem. 0, Yes. (2) Rem. 1, No. (3) Rem. -72 , No.
 (4) Rem. 0, Yes. (5) Rem. 0, Yes. (6) Rem. 356, No.

Exercise 7.13

- (1) Yes, $x + 5$. (2) Yes, $7x^2 + 3x + 2$. (3) No. (4) No.
 (5) Yes, $2x^4 - 4x^3 + 5x^2 - 2x + 2$. (6) Yes, $5x^2 - 10x + 8$.
 (7) Yes, $3x^4 - 3x^3 + x^2 - x + 9$. (8) No. (9) No. (10) Yes, $x^2 + 5$
 (11) No. (12) No. (13) No. (14) Yes, $3x^3 + 2x - 1$.
 (15) Yes, $3x^3 - 3x^2 - 8x + 8$. (16) $c = 20$. (17) $p = 66$. (18) $a = 6$.
 (19) $p = -5$. (20) $a = 5$. (21) $a = 5$. (22) $a = 25$. (23) $p = 16$. (24) $a =$
 (25) $p = q = -14$.

Exercise 7.14

- (1) Yes, $x^2 + 4x - 1$. (2) No. (3) Yes, $x^3 - 2x^2 + 2x + 8$.
 (4) Yes, $4x^3 - 3x^2 - 8x + 3$. (5) Yes, $7x^4 - 2x^3 + 6x^2 + 9x - 2$.
 (6) Yes, $5x^3 + 7x^2 + 7x - 2$.

Exercise 7.15

- (1) Yes, $x^2 + 2x - 6$. (2) No. (3) Yes, $x^3 - 4x^2 + 6x + 1$.
 (4) Yes, $3x^3 - 5x^2 + 9x + 3$. (5) Yes, $6x^4 + 3x^3 + 2x^2 + 5x$.
 (6) Yes, $7x^3 - 7x^2 - 2x + 7$.

Exercise 7.16

- | | |
|--------------------------------------|--------------------------------------|
| (1) $(x - 1)(x + 3)(x + 5)$. | (2) $(x + 1)(x + 2)(x + 3)$. |
| (3) $(x - 1)(x + 2)(x - 3)$. | (4) $(x - 1)^2(x + 2)$. |
| (5) $(x - 1)(x - 2)(x + 3)$. | (6) $(x + 2)(x - 4)^2$. |
| (7) $(x + 2)(x - 7)(x + 5)$. | (8) $(x - 2)(x - 3)(x + 5)$. |
| (9) $(x - 1)(x + 2)(x + 12)$. | (10) $(x + 1)^2(x + 2)$. |
| (11) $(x - 1)(x + 1)(x - 2)$. | (12) $(x - 1)(x + 2)(x - 4)$. |
| (13) $(x + 1)(x + 3)(x + 5)$. | (14) $(x + 1)(x - 2)(x + 7)$. |
| (15) $(x + 1)(x - 3)(x + 4)$. | (16) $(x - 2)(x + 3)(x + 5)$. |
| (17) $(x - 2)(x - 3)(x - 4)$. | (18) $(x - 3)(x - 5)(x - 6)$. |
| (19) $(x - 2)(x - 4)(x + 6)$. | (20) $(x + 4)(x + 5)(x - 9)$. |
| (21) $(x - 3)(x - 4)(x + 7)$. | (22) $(x - 3)(x - 5)(x + 8)$. |
| (23) $(x - 3)(x - 4)(x - 7)$. | (24) $(x - 3)^2(x - 4)$. |
| (25) $(x - 1)(x + 1)(x^2 + x + 3)$. | (26) $(x - 1)(x + 2)(x^2 + x + 5)$. |
| (27) $(x - 1)(x + 2)(x - 3)^2$. | (28) $(x + 1)(x - 2)^2(x - 3)$. |

- (29) $(x-1)(x+1)(x+2)(x+3)$.
 (30) $(x-1)(x+1)^2(x-2)$.
 (31) $(x-1)^2(x+1)(x-2)$.
 (32) $(x-1)^2(x^2+1)$.
 (33) $(x+1)(x-2)(x-3)(x-4)$.
 (34) $(x-3)(x^2+4)(x+12)$.
 (35) $(x+2)(x^2-x+4)$.
 (36) $(x-1)(x-9)(x-12)$.
 (37) $(x+3)(x^2-2x+9)$.
 (38) $(x-3)(x-7)(x+10)$.
 (39) $(x+2)(x+3)(x+6)$.
 (40) $(x+1)^3(x-2)$.
 (41) $(x-1)(x+1)(x-3)^2$.
 (42) $(x+1)(x-2)(x^2+2x+3)$.
 (43) $(x+2)(x-3)(x^2-x+5)$.
 (44) $(x-2)(x+2)(x^2+2x+2)$.
 (45) $(x-1)(x+1)(x^2+4x+7)$.

Exercise 7.17

- (1) $(x-1)^2(x-2)^2$. (2) $(x+2)^4(x-2)(x+5)^2$. (3) $(x+3)(x-4)^2$.
 (4) $(x-2)^4$, (5) $x^2(x+2)(x-3)^2$.

Exercise 7.18

- (1) $(x-1)^3(x-2)^2(x+3)$. (2) $(x+2)^5(x-2)^2(x+5)^3$.
 (3) $(x+3)^2(x-4)^3(x-2)^2(x+4)$.
 (4) $(x+1)(x+2)(x-2)^5(x+3)^4(x+5)$.
 (5) $x^3(x-1)^4(x+2)^3(x-3)^2$.

Exercise 7.19

- | G.C.D. | L.C.M. |
|-------------------|--------------------------------------|
| (1) $x+3$ | $(x+3)(x+4)(x^2+x+1)$. |
| (2) $x-1$ | $(x-1)(x-2)(x-3)(x^2+x+1)$. |
| (3) x^2+x+2 | $(x-1)(x^2+x+2)(x^2-x+2)$. |
| (4) $2x-5$ | $(2x-5)(2x+5)(3x-4)$. |
| (5) $(x+1)(x+2)$ | $(x+1)(x+2)(x-1)(x-2)$. |
| (6) $(x+1)(2x-1)$ | $(x+1)(2x-1)(x-1)(2x+1)(2x^2+x+1)$. |
| (7) 1 | $(x+2)(x+3)(x+4)(x+5)$. |
| (8) $x+4$ | $(x+4)(x+3)(x+5)(x^2-4x+16)$. |
| (9) x^2-x+1 | $(x^2-x+1)(x+1)(x^2+x+1)(x^2-x-1)$. |
| (10) $x-2$ | $x(x-2)(2x-1)(3x-2)(x^2+2x+4)$. |
| (11) $x(x+5)$ | $4x^2(x+5)(x-1)(3x-1)(x^2-5x+25)$. |

- | | |
|-----------------------|---|
| (12) $x - 7$ | $3x^2(x - 7)(x + 4)(x + 7)(x - 2).$ |
| (13) $2x(x - 3)$ | $36x^2(x - 3)(x - 4)(x + 3)(x^2 + 3x + 9).$ |
| (14) 1 | $(x + 4)(x - 6)(x^2 - 6).$ |
| (15) $x(x - 3)$ | $105x^2(x - 3)^2(x + 3)(x^2 + 3x + 9).$ |
| (16) $x - 2$ | $x(x - 2)(x + 2)(2x - 1)(3x - 2).$ |
| (17) $3x(2x + 3)$ | $18x^2(2x + 3)(2x - 3)(3x - 2)(4x^2 - 6x + 9).$ |
| (18) $3x(x - 3)$ | $18x^3(x - 1)(x - 3)(x - 4)(x - 5).$ |
| (19) $x + 5$ | $12(x - 5)(x + 5)(x - 10)(x^2 - 5x + 25).$ |
| (20) $3(x - 3)$ | $18x(x - 3)(x + 3)(x + 5)(x - 8).$ |
| (21) $2x - 3$ | $6x(2x - 3)(2x + 3)(x + 1)(2x + 1).$ |
| (22) $x + 7$ | $(x + 4)(x + 7)(x - 8)(x^2 - 7x + 49).$ |
| (23) $x - 7$ | $6x^2(x - 7)(x + 4)(x + 7)(x - 2).$ |
| (24) $x + 7$ | $2(x + 7)(x - 7)(3x - 5)(x^2 - 7x + 49).$ |
| (25) $x(x + 2)$ | $6x(x + 2)(x - 2)(x - 3)(x^2 - 2x + 4).$ |
| (26) $x(2x + 3)$ | $15x^2(x - 2)(2x + 3)(2x - 3)(4x^2 - 6x + 9).$ |
| (27) $a(x - 2)$ | $6ax(x - 2)(x + 2)(x - 3)(x^2 + 2x + 4).$ |
| (28) $2(x - 2)$ | $24x(x - 2)(x - 3)(x - 5)(x + 5).$ |
| (29) $x + 3$ | $x(x - 2)(x + 2)(x - 3)(x + 3)(x - 6).$ |
| (30) $(x - 2)(x + 3)$ | $(x - 1)(x + 1)(x - 2)(x + 3)(x^2 + 2x + 4).$ |
| (31) $(x + 3)(x + 5)$ | $(x + 1)(x - 1)(x + 3)(x + 5)(x - 8).$ |
| (32) $(x + 1)(x + 1)$ | $(x + 1)(x + 2)(x - 1)(x - 2)(x - 3)(x - 4).$ |
| (33) $x^3 + 2x + 4$ | $2(x - 2)(x + 3)(x^2 + 2x + 4)(x^2 - 2x + 4).$ |
| (34) $x^2 - 2x + 2$ | $2(x + 2)(x^2 - 2x + 2)(x^2 - 2x - 2)(x^2 + 2x + 2).$ |
| (35) $x - 2$ | $5(x - 2)(x + 2)(x - 3)(x + 3)(x^2 + 2x + 4).$ |
| (36) $(x - 2)(x + 2)$ | $3(x - 1)(x + 1)(x - 2)(x + 2)(x - 5)(x + 5)(x^2 + 4).$ |
| (37) $3x + 1$ | $(3x + 1)^3(29x - 7)(9x^2 - 3x + 1).$ |
| (38) $x + 2$ | $(x - 2)(x + 2)(x^2 - 2x + 4)(x^2 + 2x + 4).$ |
| (39) $x^2 + x + 1$ | $x(x - 1)(x^2 - x + 1)(x^2 + x + 1).$ |
| (40) $x^2 - 1$ | $(x^2 - 1)(x^2 + 1)(x^4 + 1)(x^4 + x^2 + 1).$ |

Exercise 8.1

- | | | | |
|-------------------------|------------------------------|-----------------------------|--|
| (1) $\frac{2yz^2}{3x}.$ | (2) $\frac{2xa^4b^3}{3yz}.$ | (3) $\frac{3x}{x + 2y}.$ | (4) $\frac{a - b}{a + b}.$ |
| (5) $\frac{1}{1 + a}.$ | (6) $\frac{x + 3y}{x - 3y}.$ | (7) $\frac{x - 7}{2x - 5}.$ | (8) $\frac{(5x + 6y)}{x(x^2 + 2xy + 4y^2)}.$ |

- (9) $\frac{a+4}{a^2+3a+14}$. (10) $\frac{1}{x}$. (11) $\frac{x-1}{x-3}$. (12) $\frac{a}{a+3}$.
 (13) $\frac{x-2}{x+2}$. (14) $\frac{x-4}{x-1}$. (15) 1. (16) $\frac{x+2}{x-3}$.
 (17) $\frac{x+3y}{x+y}$. (18) 1. (19) $\frac{2x-1}{(x-1)(2x-3)}$. (20) $\frac{1}{x-3}$.
 (21) $\frac{1}{x-8}$. (22) $\frac{x^2}{x+y}$. (23) $\frac{x^2(x+y)}{(x-y)^2}$. (24) $\frac{2x+5}{(x+3)(x+2)}$.
 (25) $\frac{-16x+59}{x^2-7x+12}$. (26) $\frac{x}{x^2-4}$. (27) $\frac{3x}{x^2-9}$. (28) $\frac{10}{a^2+6a+8}$.
 (29) $\frac{1}{1+x}$. (30) $\frac{-5y}{(x+2y)(x-3y)}$.

Exercise 8.2

- (1) 3. (2) $\frac{7}{18}$. (3) $\frac{3x+2}{10x}$. (4) $\frac{7}{3(x+4)}$. (5) $\frac{21}{x+4}$.
 (6) $\frac{2(3x+2)}{5x}$. (7) $\frac{x+4}{(x-2)(2x-1)}$. (8) $\frac{(x+4)(2x-1)}{x-2}$.
 (9) $\frac{(7x+5)(5x-4)}{2x-3}$. (10) $\frac{x^2+1}{x^2-1}$. (11) $\left(\frac{x+1}{x-1}\right)^2$. (12) $\left(\frac{2+1}{x-1}\right)^3$.
 (13) x . (14) x . (15) x . (16) x . (17) $\frac{1}{x(x+h)}$.
 (18) $\frac{-(2x+h)}{x^2(x+h)^3}$. (19) $\frac{x-1}{x+5}$. (20) b . (21) $\frac{1}{2a}$. (22) 2. (23) $-\frac{1}{x}$.
 (24) x . (25) x . (26) x . (27) $\frac{1}{100}$. (28) $\frac{1}{100}$. (29) $\frac{y(-1)}{x(x+1)}$. (30) -1 .

Exercise 8.3

- (1) $\{11\}$. (2) $\{6\}$. (3) $\{1\}$. (4) $\{2\}$. (5) $\{20\}$. (6) $\{4\}$.
 (7) $\{4\}$. (8) $\{14\}$. (9) $\{3\}$. (10) $\{4\}$. (11) $\{\frac{3}{2}\}$. (12) $\{4\}$.

Exercise 8.4

- (1) $\{-2\}$. (2) $\{\frac{25}{7}\}$. (3) $\{-\frac{8}{3}\}$. (4) $\{9\}$. (5) $\{\frac{7}{3}\}$.
 (6) $\{-\frac{3}{2}\}$. (7) $\{\frac{17}{3}\}$. (8) $\{-6\}$. (9) $\{-\frac{2}{13}\}$. (10) $\{\frac{11}{4}\}$.

Exercise 8.5

- (1) $\{5\}$. (2) $\{7\}$. (3) $\{7\}$. (4) $\{5\}$. (5) $\{-\frac{3}{4}\}$.
 (6) $\{1\}$. (7) $\{4\}$. (8) $\{-\frac{3}{8}\}$. (9) $\{7\}$. (10) $\{\frac{9}{5}\}$.

Exercise 8.6

- (1) $\{5\}$. (2) $\{-6\}$. (3) $\{5\}$. (4) $\{-\frac{3}{4}\}$. (5) $\{0\}$.
 (6) $\{-\frac{5}{11}\}$. (7) $\{\frac{7}{2}\}$. (8) $\{7\}$. (9) $\{4\}$. (10) $\{\frac{1}{8}\}$.

Exercise 8.7

- (1) $\{-6\}$. (2) $\{-\frac{7}{2}\}$. (3) $\{4\}$. (4) $\{9\}$. (5) $\{\frac{3}{2}\}$.
 (6) $\{7\}$. (7) $\{-\frac{3}{2}\}$. (8) $\{8\}$. (9) $\{\frac{5}{2}\}$. (10) $\{6\}$.
 (11) $\{\frac{13}{2}\}$. (12) $\{4\}$. (13) $\{4\}$. (14) $\{-5\}$. (15) $\{5\}$.

Exercise 9.3

- (1) $(0, 0)$. (2) $(0, -2)$. (3) $(0, 2)$. (4) $(0, 3)$. (5) $(0, 3)$.
 (6) The point $(3, -18)$ lies on the graph of $y = -2x^2$ and its symmetric point on the graph is $(-3, -18)$.
 (7) $(2.4, 6), (-2.4, 6)$. The graph of $y = x^2$ lies in the first and the second quadrant while the graph of $y = -8$ lies in the third and the fourth quadrant. Hence the graph of $y = -8$ does not cut the graph of $y = x^2$.
 (8) 2.0 or -2.0 , 2.6 or -2.6 , 3.2 or -3.2 .

Exercise 9.4

- (1) $2 \cdot 5$. (2) $-2 \cdot 5$. (3) $-2 \cdot 8$. (4) $-1 \cdot 7$. (5) $2 \cdot 7$. (6) $2 \cdot 9$. (7) $-2 \cdot 9$.
 (8) $-1 \cdot 4$.

Exercise 9.5

- (1) $(3, 0), x = 3$. (2) $(-2, 0), x = -2$. (3) $(2, 0), x = 2$.

Exercise 9.6

- (2) $(-2, 0), x = 2$. (3) in the 3rd and 4th quadrants. $(2, 0), x = 2$. (4) $(1, -6); x = 1$.
 (5) The equation $y = x^2 - 4x + 5$ can be transformed as $y = (x-2)^2 + 1$. As the values of y are positive and the vertex of the curve is $(2, 1)$ the curve will not cut the X-axis. (6) $(3, 0), (-1, 0)$. (7) $x = 3$ and $x = -1$. The x-co-ordinates of the points of intersection where the curve intersects the X-axis and the values of x in the solution set of $x^2 - 2x - 3 = 0$ are correspondingly equal.

Exercise 9.7

- (1) $\{5.2, -1.2\}$, (2) $\{3.7, -0.3\}$, (3) $\{1.2, -3.2\}$, (4) $\{4, -1\}$
 (5) $\{4.2, -1.2\}$.

Exercise 9.8

- (1) $\{3, -1\}$, (2) $\{5, -1\}$, (3) $\{7, -1\}$.
 (4) $\left\{\frac{1+\sqrt{61}}{6}, \frac{1-\sqrt{61}}{6}\right\}$, (5) $\{4, -1\}$,
 (6) 9, 12 or $-9, -12$. (7) 15 and 8 c.m. (8) 7 and 15 cm.
 (9) 67.5 km. p.h. (10) 45.

Exercise 9.9

- (1) The values of x in the solution set are equal.
 (2) The values of x in the solution set are distinct.
 (3) The values of x in the solution set are equal.
 (4) The solution set is empty.
 (5) The values of x in the solution set are distinct.
 (6) The solution set is empty.
 (7) The solution set consists of two distinct numbers.
 (8) The solution set consists of two distinct numbers.
 (9) The solution set consists of two equal numbers.
 (10) The solution set consists of two distinct numbers.
 (11) The solution set is empty.
 (12) $\left\{\frac{1+\sqrt{37}}{2}, \frac{1-\sqrt{37}}{2}\right\}$, irrational.
 (13) $\{3+\sqrt{5}, 3-\sqrt{5}\}$, irrational.
 (14) $\left\{\frac{3}{2}, -2\right\}$, rational.
 (15) $\{-2+\sqrt{3}, -2-\sqrt{3}\}$, irrational.
 (16) $\left\{\frac{1}{2}, -\frac{1}{3}\right\}$ rational,
 (17) $\left\{\frac{-4+\sqrt{5}}{2}, \frac{-4-\sqrt{5}}{2}\right\}$, irrational.

Exercise 10.1

- (1) (i) $1:3$, (ii) $2:7$, (iii) $3:5$, (iv) $2:7$, (v) $11:40$,
(vi) $7:100$, (vii) $x:2y$, ($x, y > 0$), (viii) $2:21$, (ix) $1:80$,
(x) $1:\sqrt{2}$, (xi) $4:\sqrt{3}$, (xii) $x\sqrt{2}:y^2$ ($x > 0, y \neq 0$).
(2) 21 years, (3) 135, 225, (4) $\frac{xa}{6}$ (5) 12 and 16 years (6) 35, 71,
(7) $25:9$, (8) $3:2; 36 \text{ cm.}$ (9) 7 cm. and 4.2 cm. (10) 140,
(11) AB and DC. (12) 86.625 sq. cm. (13) 21%.

Exercise 10.2

- (1) (i) Commensurable ratios. (i) (iii) Incommensurable ratios (ii), (iv), (v)
(vi) If x is rational then the ratio is commensurable otherwise incommensurable.
(2) (i) $<$, (ii) $<$, (iii) $<$, (iv) $=$ (v) $>$, (vi) $<$.
(3) $x\sqrt{2}$, $\sqrt{2}:1$ It is incommensurable.
(4) (i) $\frac{5}{7}:1$, (ii) $\frac{28}{27}:1$, (iii) $2\sqrt{3}:1$, (iv) $\sqrt{2}:1$, (v) $\sqrt{3}:1$.

Exercise 10.3

- (1) All the statements are true.
(2) (i) 10, 20. (ii) 30, 18. (iii) 34, 54, 18, 2, (iv) 6, 5, 35, 111.
(3) (i) 3, (ii) 3, (iii) 9, (iv) 35, (v) $\frac{20}{3}$.
(4) (i) $\frac{3}{7}, \frac{y}{7}$. (ii) $\frac{5}{a}, \frac{3}{5}$. (iii) $\frac{1}{3}, \frac{2}{3}$. (iv) $\frac{c}{d}, \frac{b}{d}$. (v) $\frac{2c}{3d}, \frac{2b}{3a}$.

Exercise 10.4

- (1) (i) 24, (ii) 21, (iii) 15, (iv) $\frac{c^2d}{a}$, (v) $12ab^3$.
(2) $\frac{3}{5}$. (3) (i) $\frac{20}{15}, \frac{4}{20}$. (ii) $5 \times 99; \frac{5}{45}$.
(iii) $\frac{63+27}{27}, 7-3$. (iv) $51-39, 11$. (v) $\frac{28}{11}, \frac{6}{11}$.
(vi) $\frac{8}{2}$, (vii) $\frac{55}{10}, \frac{65}{45}$. (viii) $\frac{3}{5}, \frac{8}{2}$. (ix) $\frac{7}{23}$, (x) $\frac{4}{13}$.
(4) (i) $\frac{7}{1}$, (ii) $\frac{5}{9}$, (iii) $\frac{2}{1}$. (iv) $\frac{7}{3}$. (5) (i) $\{2\}$, (ii) $\{0, 1\}$. (6) 2.

Exercise 10.5

- (1) (i) $ab - \frac{1}{ab}$ or $\frac{1}{ab} - ab$, (ii) $\pm \frac{2a}{b}$, (iii) $\pm (a^2 - b^2)$, (iv) y^2 ,
(v) 2, (vi) $x = 16$, $y = 64$, (vii) $x = 2$, $y = 18$.
(5) 3. (6) 1, 3, 9. (7) 2, 16, 128. (8) 2, 4, 8, 16, 32.

Exercise 10.7

- (1) (i) 1, 36. (ii) 18, 7, $x + y + z$. (iii) 18.
(iv) $2b + 3d - 5f$, $5a + 7c - 11e$. (v) $x + y$. (10) {9}.

Exercise 11.1

(1) to (5) are functions :

(6) Not a function. The element 'a' in the first set is associated with more than one element of the second set. Hence the 2nd condition is not satisfied.

(7) and (8) are functions.

(9) Not a function. The element 'Shri Tope' in the first set has no element in the second set associated with it. Hence the first condition is not satisfied. Moreover, the element 'Shri Joshi' and 'Shri Mehta' have two elements each from the second set associated with them. Hence, the second condition is also violated.

(10) Not a function. The element 'Sadhana' in the first set has no corresponding element in the second set. Hence the first condition is violated.

(11) Not a function. Both the conditions are violated.

Exercise 11.2

Functions (1), (2), (3), (4), (5), (8), (10), (11), (12).

Not functions (6), (7), (9).

- (13) (i) 0, (ii) $\frac{3}{4}$, (iii) 2, (iv) $2 + \sqrt{2}$, (v) 0,
(vi) 2, (vii) $\frac{15}{4}$, (viii) $5 - 3\sqrt{3}$.

- (15) (i) {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)}.
(ii) {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}.
(iii) {(1, 4), (2, 4), (3, 4), (4, 4), (5, 4)}.
(iv) {(1, -1), (2, -2), (3, -3), (4, -4), (5, -5)}.
(v) {(1, 1), (2, $\sqrt{2}$), (3, $\sqrt{3}$), (4, 2), (5, $\sqrt{5}$)}.

- (vi) $\left\{ \left(1, \frac{1}{2}\right), \left(2, \frac{1}{4}\right), \left(3, \frac{1}{8}\right), \left(4, \frac{1}{16}\right), \left(5, \frac{1}{32}\right) \right\}$

- (17) (i) $x > 2$ and $x < -2$; (ii) $x = 4$; (iii) $x = -2$;
 (iv) The function is defined for all real numbers. (v) $x = -7$ and $x = 1$.

(18) $\frac{1}{4}, 2, \frac{1}{\sqrt{2}}, \sqrt{2}$.

- (19) (a) $\{(2, 1), (4, 1)\}$; (b) $\{(1, \frac{1}{2})\}$; (c) $\{\}$ (empty set).

- (20) (i) 0, (ii) 1, (iii) 2, (iv) 3, (v) 2,
 (vi) No value as the function is not defined.

Exercise 12.1

- (1) (i) -5, (ii) $\frac{9}{4}$, (iii) 1.2, (iv) 0.7.

- (2) (i) Yes. Yes. The ratio $y : x$ is 2:3 a constant. (ii) Yes, $\frac{3}{4}$.

(iii) $-2, -\frac{9}{2}, 4, 1.4, 1.8$.

- (3) (i) $1, x = y$. (ii) $2, y = 2x$. (iii) $-\frac{3}{2}, y = -\frac{3}{2}x$.

(iv) $\frac{1}{2}, y = \frac{1}{2}x$. (v) $-\frac{5}{2}, y = -\frac{5}{2}x$. (vi) $\frac{22}{7}, y = \frac{22}{7}x$.

(4) $24, 11, \frac{5}{3}, y = 3x$.

(5) $y = \frac{5}{2}x; \frac{5}{2}, 4, \frac{25}{2}, 0, 6$.

- (6) (i) $c = \text{circumference}$ $r = \text{radius}$, $c \propto r$. (ii) $s = \text{side}$, $h = \text{height}$, $g \propto s$.
 (iii) $V = \text{Volume}$, $r = \text{radius}$, $v \propto r^3$. (iv) $t = \text{Time}$, $d = \text{mean distance}$, $t \propto d$.

(7) $y = \frac{5}{3}x$, (i) 3.75, (ii) 3.

(8) (i) $\frac{32}{3}$, (ii) $\frac{2}{3}$.

(9) 616 sq. cm. (10) 18 meters. (11) Rs. 784.

Exercise 12.2

- (1) (i) 1, 48, $\frac{1}{4}, \frac{1}{8}$. (ii) 12, 225. (iii) 8, 5, 240
 (iv) 6, 18, 324, 432.

(2) 15, -10, 6, -5, 60, $-\frac{1}{2}$.

- (3) (i) $6, xy = 6$; (ii) $\frac{3}{2}, xy = \frac{3}{2}$; (iii) $5, xy = 5$; (iv) $-12, xy = -12$;
 (v) $-7, xy = -7$; (vi) $21, xy = 21$.
- (4) (i) y varies directly as x ; (ii) y varies inversely as the square of x ;
 (iii) y varies inversely as the positive square root of x .
- (5) (i) $xy = 10$, (ii) 20 , (iii) 50 , (iv) The value of y will be $\frac{1}{2}$ as much.
- (6) (i) 2 , (ii) The value of y will be $\frac{1}{4}$ as much
- (7) $\frac{1}{64}$.
- (10) 6 amperes. (11) 4 . (12) $\frac{9}{2}$ gm. wt.

Exercise 12.3.

- (1) 9 cm. (2) 3 sq. m. (3) 20.7 ohms. (4) 18 cm. (5) 3.3 gm.
 (6) 85 sq. cm. (7) 299 c.c. (8) 18 seconds
 (9) approx. 102 cm. (10) 10 .

Exercise 12.4

- (1) 36 . (2) 13 . (3) 6 . (4) Rs. 4555 .
 (5) Rs. 540 . (6) Rs. 412 . (7) Rs. 8150 . (8) Rs. 1360 .